Pensieve header: The WG Algebra with testing.

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In[∘]:=
```
DeclareGroup[S_k_] := Module[{α, β, e, γs},
  Clear[G, n, g, ι, m, inv];
  G = PermutationCycles /@ (Permutations@Range@k);
  n = Length[G];
  Do[g[α] = e = G[[α]]; ι[e] = α, {α, n} ];
  m[] = ι[Cycles[{}]];
  Do[m[α, β] = ι[g[α] ~PermutationProduct~ g[β]], {α, n}, {β, n} ];
  m[α_] := α; m[α_, β_, γs__] := m[m[α, β], γs];
  Do[inv[α] = ι[InversePermutation[g[α]]], {α, n}]
 ]
```

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In[∘]:=
```
DeclareGroup[S3];
Table[m[i, j], {i, n}, {j, n}] // MatrixForm
```

Out[∘]//MatrixForm=

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$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 5 & 1 & 6 & 2 & 4 \\ 4 & 6 & 2 & 5 & 1 & 3 \\ 5 & 3 & 6 & 1 & 4 & 2 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix}$$

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In[∘]:=
```
Basis[] = {1};
Basis[i_, is___] := Flatten@Table[W_i[α, β] Basis[is], {α, n}, {β, n}]
```

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In[∘]:=
```
Basis[1, 2]
```

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Out[∘]=
```
{W1[1, 1] W2[1, 1], W1[1, 1] W2[1, 2], W1[1, 1] W2[1, 3], W1[1, 1] W2[1, 4], W1[1, 1] W2[1, 5],
 W1[1, 1] W2[1, 6], W1[1, 1] W2[2, 1], W1[1, 1] W2[2, 2], W1[1, 1] W2[2, 3], … 1278 …,
 W1[6, 6] W2[5, 4], W1[6, 6] W2[5, 5], W1[6, 6] W2[5, 6], W1[6, 6] W2[6, 1], W1[6, 6] W2[6, 2],
 W1[6, 6] W2[6, 3], W1[6, 6] W2[6, 4], W1[6, 6] W2[6, 5], W1[6, 6] W2[6, 6]}
```

large output | show less | show more | show all | set size limit...

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In[∘]:=
```
m_{i_,j_→k_}[ℰ_] :=
  Expand[ℰ] /. W_i[α_, β_] W_j[γ_, δ_] :→ If[m[α, β] == m[β, γ], W_k[α, m[β, δ]], 0];
η_i_[ℰ_] := Expand[ℰ Sum[W_i[α, m[]], {α, n}]];
```

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In[∘]:=
```
Δi_→j_,k_[ℰ_] := Expand[ℰ /. Wi[α_, β_] :→ Sum[Wj[γ, β] Wk[m[α, inv[γ]], β], {γ, n}]];
εi_[ℰ_] := Expand[ℰ /. Wi[α_, β_] :→ If[α == m[], 1, 0]];
```

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In[∘]:=
```
Si_[ℰ_] := Expand[ℰ /. Wi[α_, β_] :→ Wi[m[inv[β], inv[α], β], inv[β]]];
```

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In[∘]:=
```
Ri_,j_ := Sum[Wi[α, m[]] Wj[β, α], {α, n}, {β, n}];
R̄i_,j_ := Sum[Wi[α, m[]] Wj[β, inv@α], {α, n}, {β, n}];
```

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In[∘]:= b = Basis[1, 2, 3]; (b // $m_{1,2\to1}$ // $m_{1,3\to1}$) == (b // $m_{2,3\to2}$ // $m_{1,2\to1}$)

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Out[∘]= True

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In[∘]:= b = Basis[1]; (b // η_2 // $m_{1,2\to1}$) == b == (b // η_2 // $m_{1,2\to1}$)

pdf

Out[∘]= True

pdf

In[∘]:= b = Basis[1]; (b // $\Delta_{1\to1,2}$ // $\Delta_{2\to2,3}$) == (b // $\Delta_{1\to1,3}$ // $\Delta_{1\to1,2}$)

pdf

Out[∘]= True

pdf

In[∘]:= b = Basis[1]; (b // $\Delta_{1\to1,2}$ // ε_2) == b == (b // $\Delta_{1\to2,1}$ // ε_2)

pdf

Out[∘]= True

pdf

In[∘]:= b = Basis[1, 2]; (b // ε_1 // ε_2) == (b // $m_{1,2\to1}$ // ε_1)

pdf

Out[∘]= True

pdf

In[∘]:= b = Basis[1, 3]; (b // $\Delta_{1\to1,2}$ // $\Delta_{3\to3,4}$ // $m_{1,3\to1}$ // $m_{2,4\to2}$) == (b // $m_{1,3\to1}$ // $\Delta_{1\to1,2}$)

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Out[∘]= True

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In[∘]:= b = Basis[1]; (b // S_1 // S_1) == b

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Out[∘]= True

pdf

In[∘]:= b = Basis[1]; (b // $\Delta_{1\to1,2}$ // S_2 // $m_{1,2\to1}$) == (b // ε_1 // η_1) == (b // $\Delta_{1\to1,2}$ // S_1 // $m_{1,2\to1}$)

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Out[∘]= True

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In[∘]:= $\left(R_{1,2}\,\overline{R}_{3,4}\ //\ m_{1,3\to1}\ //\ m_{2,4\to2}\right) == (1\ //\ \eta_1\ //\ \eta_2) == \left(R_{1,2}\,\overline{R}_{3,4}\ //\ m_{1,3\to1}\ //\ m_{4,2\to2}\right)$

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Out[∘]= True

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In[∘]:= $(R_{1,2}\,R_{4,3}\,R_{5,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3}) == (R_{2,3}\,R_{1,4}\,R_{5,6}\ //\ m_{1,5\to1}\ //\ m_{2,6\to2}\ //\ m_{3,4\to3})$

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Out[∘]= True

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In[∘]:= $\{(R_{1,3}\ //\ \Delta_{1\to1,2}) == (R_{2,3}\,R_{1,4}\ //\ m_{3,4\to3}),\ (R_{1,2}\ //\ \Delta_{2\to2,3}) == (R_{0,2}\,R_{1,3}\ //\ m_{0,1\to1})\}$

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Out[∘]= {True, True}

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In[∘]:= $\{(R_{1,2}\ //\ \varepsilon_1) == (1\ //\ \eta_2),\ (R_{1,2}\ //\ \varepsilon_2) == (1\ //\ \eta_1)\}$

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Out[∘]= {True, True}

pdf

In[∘]:= $(R_{1,2}\ //\ S_1) == \overline{R}_{1,2} == (R_{1,2}\ //\ S_2)$

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Out[∘]= True

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Does R1 hold?

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In[∘]:= $\{R_{1,2}\ //\ m_{1,2\to1},\ 1\ //\ \eta_1\}$

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Out[∘]= $\{W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 4] + W_1[5, 5] + W_1[6, 6],$
$W_1[1, 1] + W_1[2, 1] + W_1[3, 1] + W_1[4, 1] + W_1[5, 1] + W_1[6, 1]\}$

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In[∘]:=
```
Ks = {PD[X[1, 4, 2, 5], X[3, 6, 4, 1], X[5, 2, 6, 3]],
   PD[X[4, 2, 5, 1], X[8, 6, 1, 5], X[6, 3, 7, 4], X[2, 7, 3, 8]],
   PD[X[1, 6, 2, 7], X[3, 8, 4, 9], X[5, 10, 6, 1], X[7, 2, 8, 3], X[9, 4, 10, 5]],
   PD[X[1, 4, 2, 5], X[3, 8, 4, 9], X[5, 10, 6, 1], X[9, 6, 10, 7], X[7, 2, 8, 3]]};
```

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In[∘]:=
```
Z[pd_PD] := Module[{z},
  z = Expand[Times @@ pd /. x : X[i_, j_, k_, l_] :> If[PositiveQ@x, R_{l,i}, R̄_{j,i}]];
  Do[z = z // m_{1,k→1}, {k, 2 Length@pd}];
  z]
```

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In[∘]:=
```
Table[K → Echo[Timing[Z[K]]], {K, Ks}]
```

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» $\{1.01563, W_1[1, 1] + 3\,W_1[2, 2] + 3\,W_1[3, 3] + W_1[4, 1] + W_1[5, 1] + 3\,W_1[6, 6]\}$

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Out[∘]= $Aborted