Pensieve header: The category GDO and the Heisenberg algebra. Based on AlexanderFromHeisenberg.nb at pensieve://Talks/LearningSeminarOnCategorification-2006/.

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Classes\\21-1350-KnotTheory"];
<< Common.m
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

A hack to fix how Expand acts on Series:

In[∘]:=
```
Unprotect[SeriesData];
SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
Protect[SeriesData];
```

In[∘]:=
```
{p*, x*, π*, ξ*} = {π, ξ, p, x}; (u_i_)* := (u*)_i; l_List* := #* & /@ l;
```

In[∘]:=
```
p*
```

Out[∘]= π

In[∘]:=
```
p_7*
```

Out[∘]= π_7

In[∘]:=
```
{p, ξ_6}*
```

Out[∘]= $\{\pi, x_6\}$

In[∘]:=
```
(𝔾_{A1_→B1_}[L1_] // 𝔾_{A2_→B2_}[L2_]) /; (B1* === A2) := Module[{b, r, x, ξ},
  r = Expand[(L1 /. Table[x → b[x], {x, B1}]) (L2 /. Table[ξ → b[ξ], {ξ, A2}])];
  Do[r = r /. b[x]^p_. b[x*]^p_. → p! /. b[x | x*] → 0, {x, B1}];
  𝔾_{A1→B2}[r]
 ]
```

In[∘]:=
```
𝔾_{{ξ_1}→{x_1}}[1 + ξ_1 x_1 + ξ_1^2 x_1^2] // 𝔾_{{ξ_1}→{x_2}}[1 + ξ_1 x_2 + ξ_1^2 x_2^2]
```

Out[∘]= $\mathbb{G}_{\{\xi_1\}\to\{x_2\}}\left[1 + x_2\,\xi_1 + 2\,x_2^2\,\xi_1^2\right]$

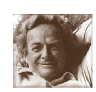
GDO := The category with objects finite sets and

$$\text{mor}(A \to B) = \left\{\mathcal{L} = \omega e^{Q}\right\} \subset \mathbb{Q}[[\zeta_A, z_B]],$$

where: • ω is a scalar. • Q is a "small" quadratic in $\zeta_A \cup z_B$.
• Compositions: $\mathcal{L} /\!/ \mathcal{M} := \left(\mathcal{L}|_{z_i \to \partial_{\zeta_i}} \mathcal{M}\right)_{\zeta_i = 0}$.

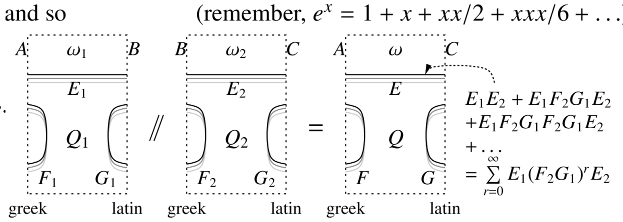
Compositions. In mor$(A \to B)$,

$$Q = \sum_{i\in A, j\in B} E_{ij}\zeta_i z_j + \frac{1}{2}\sum_{i,j\in A} F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j,$$



R. Feynman

and so (remember, $e^x = 1 + x + xx/2 + xxx/6 + \dots$)



where • $E = E_1(I - F_2G_1)^{-1}E_2$ • $F = F_1 + E_1F_2(I - G_1F_2)^{-1}E_1^T$
• $G = G_2 + E_2^T G_1(I - F_2G_1)^{-1}E_2$ • $\omega = \omega_1\omega_2 \det(I - F_2G_1)^{-1/2}$

In[∘]:=
```
𝔼s1_[ωQ1__] ≡ 𝔼s2_[ωQ2__] := s1 === s2 ∧ Simplify[{ωQ1} == {ωQ2}]
```

In[∘]:=
```
𝔼A1_→B1_[ω1_, Q1_] 𝔼A2_→B2_[ω2_, Q2_] ^:= 𝔼A1⋃A2→B1⋃B2[ω1 ω2, Q1 + Q2]
```

In[∘]:=
```
CF = ExpandNumerator@*ExpandDenominator@*PowerExpand@*Factor;
```

In[∘]:=
```
(𝔼A1_→B1_[ω1_, Q1_] // 𝔼A2_→B2_[ω2_, Q2_]) /; (B1* === A2) :=
 Module[{i, j, E1, F1, G1, E2, F2, G2, I, M = Table},
  I = IdentityMatrix@Length@B1;
  E1 = M[∂i,j Q1, {i, A1}, {j, B1}]; E2 = M[∂i,j Q2, {i, A2}, {j, B2}];
  F1 = M[∂i,j Q1, {i, A1}, {j, A1}]; F2 = M[∂i,j Q2, {i, A2}, {j, A2}];
  G1 = M[∂i,j Q1, {i, B1}, {j, B1}]; G2 = M[∂i,j Q2, {i, B2}, {j, B2}];
  𝔼A1→B2[CF[ω1 ω2 Det[I - F2.G1]^(-1/2)], CF@Plus[
     If[A1 === {} ∨ B2 === {}, 0, A1.E1.Inverse[I - F2.G1].E2.B2],
     If[A1 === {}, 0, 1/2 A1.(F1 + E1.F2.Inverse[I - G1.F2].E1ᵀ).A1],
     If[B2 === {}, 0, 1/2 B2.(G2 + E2ᵀ.G1.Inverse[I - F2.G1].E2).B2]]]]
```

In[∘]:=
```
Random𝔼A_→B_[r_] := Module[{ri},
  ri := RandomInteger[{-r, r}];
  𝔼Table[ξi,{i,A}]→Table[xj,{j,B}][
   ri,
   Sum[ri ℏ ξi ξj, {i, A}, {j, A}] +
    Sum[ri ℏ ξi xj, {i, A}, {j, B}] + Sum[ri ℏ xi xj, {i, B}, {j, B}]
  ]]
```

In[∘]:=
```
E1 = Random𝔼{1,2}→{1,2,3,4}[5]
```

Out[∘]= $\mathbb{E}_{\{\xi_1,\xi_2\}\to\{x_1,x_2,x_3,x_4\}}\left[-5,\ -3\hbar x_1 x_2 + 8\hbar x_1 x_3 + 2\hbar x_2 x_3 + 10\hbar x_2 x_4 - 3\hbar x_3 x_4 - 4\hbar x_4^2 - \hbar x_1 \xi_1 + 5\hbar x_2 \xi_1 + 2\hbar x_3 \xi_1 + 5\hbar x_4 \xi_1 - 5\hbar \xi_1^2 - \hbar x_1 \xi_2 + 5\hbar x_2 \xi_2 - 3\hbar x_3 \xi_2 - \hbar x_4 \xi_2 + 5\hbar \xi_2^2\right]$

In[∘]:=
```
E2 = Random𝔼{1,2,3,4}→{1,2,3}[5]
```

Out[∘]= $\mathbb{E}_{\{\xi_1,\xi_2,\xi_3,\xi_4\}\to\{x_1,x_2,x_3\}}\left[-3,\ -2\hbar x_1^2 - \hbar x_1 x_2 + 2\hbar x_2^2 - 7\hbar x_1 x_3 - 3\hbar x_3^2 + 4\hbar x_1 \xi_1 + 2\hbar x_2 \xi_1 - 5\hbar x_3 \xi_1 + 5\hbar \xi_1^2 + 4\hbar x_1 \xi_2 + \hbar x_2 \xi_2 - 4\hbar x_3 \xi_2 + 2\hbar \xi_1 \xi_2 - 3\hbar \xi_2^2 + 5\hbar x_1 \xi_3 - 4\hbar x_2 \xi_3 + \hbar x_3 \xi_3 + 2\hbar \xi_1 \xi_3 + 9\hbar \xi_2 \xi_3 - \hbar \xi_3^2 + 5\hbar x_1 \xi_4 + \hbar x_2 \xi_4 - 2\hbar x_3 \xi_4 - 3\hbar \xi_1 \xi_4 + 5\hbar \xi_2 \xi_4 + 6\hbar \xi_3 \xi_4 - \hbar \xi_4^2\right]$

In[∘]:= **E1 // E2**

Out[∘]= $\mathbb{E}_{\{\xi_1,\xi_2\}\to\{x_1,x_2,x_3\}}\Big[\frac{15}{\sqrt{1-136\hbar^2+14842\hbar^4-1663248\hbar^6+67366973\hbar^8}},$
$\big(-2\hbar x_1^2+449\hbar^3x_1^2-47300\hbar^5x_1^2+5367684\hbar^7x_1^2-371735337\hbar^9x_1^2-\hbar x_1x_2+125\hbar^3x_1x_2-19579\hbar^5x_1x_2+842675\hbar^7x_1x_2+16334908\hbar^9x_1x_2+2\hbar x_2^2-332\hbar^3x_2^2+44640\hbar^5x_2^2-4657172\hbar^7x_2^2+175745486\hbar^9x_2^2-7\hbar x_1x_3+675\hbar^3x_1x_3-80244\hbar^5x_1x_3+9888615\hbar^7x_1x_3-268974127\hbar^9x_1x_3+178\hbar^3x_2x_3-33870\hbar^5x_2x_3+2703550\hbar^7x_2x_3-83382386\hbar^9x_2x_3-3\hbar x_3^2+370\hbar^3x_3^2-31715\hbar^5x_3^2+4497644\hbar^7x_3^2-238889316\hbar^9x_3^2+51\hbar^2x_1\xi_1-5441\hbar^4x_1\xi_1+591929\hbar^6x_1\xi_1-53218531\hbar^8x_1\xi_1+1796\hbar^4x_2\xi_1-402824\hbar^6x_2\xi_1+19354404\hbar^8x_2\xi_1-23\hbar^2x_3\xi_1-833\hbar^4x_3\xi_1+184295\hbar^6x_3\xi_1+7715361\hbar^8x_3\xi_1-5\hbar\xi_1^2+857\hbar^3\xi_1^2-108983\hbar^5\xi_1^2+12654179\hbar^7\xi_1^2-597755768\hbar^9\xi_1^2-4\hbar^2x_1\xi_2-1717\hbar^4x_1\xi_2+126890\hbar^6x_1\xi_2-7633297\hbar^8x_1\xi_2+14\hbar^2x_2\xi_2-730\hbar^4x_2\xi_2-233390\hbar^6x_2\xi_2+15445354\hbar^8x_2\xi_2-16\hbar^2x_3\xi_2+178\hbar^4x_3\xi_2+287172\hbar^6x_3\xi_2-12013022\hbar^8x_3\xi_2-171\hbar^3\xi_1\xi_2-19197\hbar^5\xi_1\xi_2+4714251\hbar^7\xi_1\xi_2-233783923\hbar^9\xi_1\xi_2+5\hbar\xi_2^2-909\hbar^3\xi_2^2+88438\hbar^5\xi_2^2-7798317\hbar^7\xi_2^2+299469331\hbar^9\xi_2^2\big)\Big/$
$\big(1-136\hbar^2+14842\hbar^4-1663248\hbar^6+67366973\hbar^8\big)\Big]$

```
In[∘]:= 𝔾[𝔼s_[ω_, Q_], d_] := 𝔾s[Expand@Series[ω Exp[Q], {ℏ, 0, d}]]
```

In[∘]:= **E1**

Out[∘]= $\mathbb{E}_{\{\xi_1,\xi_2\}\to\{x_1,x_2,x_3,x_4\}}\big[-5,\ -3\hbar x_1x_2+8\hbar x_1x_3+2\hbar x_2x_3+10\hbar x_2x_4-3\hbar x_3x_4-4\hbar x_4^2-\hbar x_1\xi_1+5\hbar x_2\xi_1+2\hbar x_3\xi_1+5\hbar x_4\xi_1-5\hbar\xi_1^2-\hbar x_1\xi_2+5\hbar x_2\xi_2-3\hbar x_3\xi_2-\hbar x_4\xi_2+5\hbar\xi_2^2\big]$

In[∘]:= **𝔾[E1, 2]**

Out[∘]=
$$
\begin{aligned}
&\mathbb{G}_{\{\xi_1,\xi_2\}\to\{x_1,x_2,x_3,x_4\}}\Big[\\
&-5+\big(15 x_1 x_2-40 x_1 x_3-10 x_2 x_3-50 x_2 x_4+15 x_3 x_4+20 x_4^2+5 x_1 \xi_1-25 x_2 \xi_1-10 x_3 \xi_1-\\
&\quad 25 x_4 \xi_1+25 \xi_1^2+5 x_1 \xi_2-25 x_2 \xi_2+15 x_3 \xi_2+5 x_4 \xi_2-25 \xi_2^2\big)\hbar+\\
&\Big(-\frac{45}{2} x_1^2 x_2^2+120 x_1^2 x_2 x_3+30 x_1 x_2^2 x_3-160 x_1^2 x_3^2-80 x_1 x_2 x_3^2-10 x_2^2 x_3^2+150 x_1 x_2^2 x_4-\\
&\quad 445 x_1 x_2 x_3 x_4-100 x_2^2 x_3 x_4+120 x_1 x_3^2 x_4+30 x_2 x_3^2 x_4-60 x_1 x_2 x_4^2-250 x_2^2 x_4^2+160 x_1 x_3 x_4^2+\\
&\quad 190 x_2 x_3 x_4^2-\frac{45}{2} x_3^2 x_4^2+200 x_2 x_4^3-60 x_3 x_4^3-40 x_4^4-15 x_1^2 x_2 \xi_1+75 x_1 x_2^2 \xi_1+40 x_1^2 x_3 \xi_1-\\
&\quad 160 x_1 x_2 x_3 \xi_1-50 x_2^2 x_3 \xi_1-80 x_1 x_3^2 \xi_1-20 x_2 x_3^2 \xi_1+125 x_1 x_2 x_4 \xi_1-250 x_2^2 x_4 \xi_1-\\
&\quad 215 x_1 x_3 x_4 \xi_1-75 x_2 x_3 x_4 \xi_1+30 x_3^2 x_4 \xi_1-20 x_1 x_4^2 \xi_1-150 x_2 x_4^2 \xi_1+115 x_3 x_4^2 \xi_1+\\
&\quad 100 x_4^3 \xi_1-\frac{5}{2} x_1^2 \xi_1^2-50 x_1 x_2 \xi_1^2-\frac{125}{2} x_2^2 \xi_1^2+210 x_1 x_3 \xi_1^2-10 x_3^2 \xi_1^2+25 x_1 x_4 \xi_1^2+125 x_2 x_4 \xi_1^2-\\
&\quad 125 x_3 x_4 \xi_1^2-\frac{325}{2} x_4^2 \xi_1^2-25 x_1 \xi_1^3+125 x_2 \xi_1^3+50 x_3 \xi_1^3+125 x_4 \xi_1^3-\frac{125 \xi_1^4}{2}-15 x_1^2 x_2 \xi_2+\\
&\quad 75 x_1 x_2^2 \xi_2+40 x_1^2 x_3 \xi_2-235 x_1 x_2 x_3 \xi_2-50 x_2^2 x_3 \xi_2+120 x_1 x_3^2 \xi_2+30 x_2 x_3^2 \xi_2+35 x_1 x_2 x_4 \xi_2-\\
&\quad 250 x_2^2 x_4 \xi_2+25 x_1 x_3 x_4 \xi_2+235 x_2 x_3 x_4 \xi_2-45 x_3^2 x_4 \xi_2-20 x_1 x_4^2 \xi_2+150 x_2 x_4^2 \xi_2-75 x_3 x_4^2 \xi_2-\\
&\quad 20 x_4^3 \xi_2-5 x_1^2 \xi_1 \xi_2+50 x_1 x_2 \xi_1 \xi_2-125 x_2^2 \xi_1 \xi_2-5 x_1 x_3 \xi_1 \xi_2+25 x_2 x_3 \xi_1 \xi_2+30 x_3^2 \xi_1 \xi_2+\\
&\quad 20 x_1 x_4 \xi_1 \xi_2-100 x_2 x_4 \xi_1 \xi_2+85 x_3 x_4 \xi_1 \xi_2+25 x_4^2 \xi_1 \xi_2-25 x_1 \xi_1^2 \xi_2+125 x_2 \xi_1^2 \xi_2-\\
&\quad 75 x_3 \xi_1^2 \xi_2-25 x_4 \xi_1^2 \xi_2-\frac{5}{2} x_1^2 \xi_2^2+100 x_1 x_2 \xi_2^2-\frac{125}{2} x_2^2 \xi_2^2-215 x_1 x_3 \xi_2^2+25 x_2 x_3 \xi_2^2-\frac{45}{2} x_3^2 \xi_2^2-\\
&\quad 5 x_1 x_4 \xi_2^2-225 x_2 x_4 \xi_2^2+60 x_3 x_4 \xi_2^2+\frac{195}{2} x_4^2 \xi_2^2+25 x_1 \xi_1 \xi_2^2-125 x_2 \xi_1 \xi_2^2-50 x_3 \xi_1 \xi_2^2-\\
&\quad 125 x_4 \xi_1 \xi_2^2+125 \xi_1^2 \xi_2^2+25 x_1 \xi_2^3-125 x_2 \xi_2^3+75 x_3 \xi_2^3+25 x_4 \xi_2^3-\frac{125 \xi_2^4}{2}\Big)\hbar^2+O[\hbar]^3\Big]
\end{aligned}
$$

In[∘]:= **𝔾[E2, 1]**

Out[∘]=
$$
\begin{aligned}
&\mathbb{G}_{\{\xi_1,\xi_2,\xi_3,\xi_4\}\to\{x_1,x_2,x_3\}}\Big[\\
&-3+\big(6 x_1^2+3 x_1 x_2-6 x_2^2+21 x_1 x_3+9 x_3^2-12 x_1 \xi_1-6 x_2 \xi_1+15 x_3 \xi_1-15 \xi_1^2-12 x_1 \xi_2-\\
&\quad 3 x_2 \xi_2+12 x_3 \xi_2-6 \xi_1 \xi_2+9 \xi_2^2-15 x_1 \xi_3+12 x_2 \xi_3-3 x_3 \xi_3-6 \xi_1 \xi_3-27 \xi_2 \xi_3+\\
&\quad 3 \xi_3^2-15 x_1 \xi_4-3 x_2 \xi_4+6 x_3 \xi_4+9 \xi_1 \xi_4-15 \xi_2 \xi_4-18 \xi_3 \xi_4+3 \xi_4^2\big)\hbar+O[\hbar]^2\Big]
\end{aligned}
$$

In[∘]:= **𝔾[E1, 1] // 𝔾[E2, 1]**

Out[∘]= $\mathbb{G}_{\{\xi_1,\xi_2\}\to\{x_1,x_2,x_3\}}\left[15+\left(-30 x_1^2-15 x_1 x_2+30 x_2^2-105 x_1 x_3-45 x_3^2-75 \xi_1^2+75 \xi_2^2\right)\hbar+O[\hbar]^2\right]$

In[∘]:= **𝔾[E1 // E2, 1]**

Out[∘]= $\mathbb{G}_{\{\xi_1,\xi_2\}\to\{x_1,x_2,x_3\}}\left[15+\left(-30 x_1^2-15 x_1 x_2+30 x_2^2-105 x_1 x_3-45 x_3^2-75 \xi_1^2+75 \xi_2^2\right)\hbar+O[\hbar]^2\right]$

In[∘]:= **𝔾[E1 // E2, 5] == (𝔾[E1, 5] // 𝔾[E2, 5])**

Out[∘]= True

In[∘]:=
$$A\_ \setminus B\_ := \text{Complement}[A, B];$$
$$\left(\mathbb{E}_{A1\_\to B1\_}[\omega 1\_, Q1\_] \,//\, \mathbb{E}_{A2\_\to B2\_}[\omega 2\_, Q2\_]\right) /; \;(B1^* =!= A2) :=$$
$$\mathbb{E}_{A1\cup(A2\setminus B1^*)\to B1\cup A2^*}[\omega 1, Q1 + \text{Sum}[\xi^*\,\xi, \{\xi, A2\setminus B1^*\}]] \;//$$
$$\mathbb{E}_{B1^*\cup A2\to B2\cup(B1\setminus A2^*)}[\omega 2, Q2 + \text{Sum}[z^*\,z, \{z, B1\setminus A2^*\}]]$$

A proof of the formula for *R* is at http://drorbn.net/cat20.

In[∘]:=
$$R_{i\_,j\_} := \mathbb{E}_{\{\}\to\{p_i,x_i,p_j,x_j\}}\left[T^{1/2}, (1-T)\,p_j\,x_j + (T-1)\,p_i\,x_i\right];$$
$$\overline{R}_{i\_,j\_} := \mathbb{E}_{\{\}\to\{p_i,x_i,p_j,x_j\}}\left[T^{-1/2}, \left(1-T^{-1}\right)p_j\,x_j + \left(T^{-1}-1\right)p_i\,x_i\right];$$
$$C_{i\_} := \mathbb{E}_{\{\}\to\{p_i,x_i\}}\left[T^{1/2}, 0\right]; \;\overline{C}_{i\_} := \mathbb{E}_{\{\}\to\{p_i,x_i\}}\left[T^{-1/2}, 0\right];$$

In[∘]:=
$$\eta_{i\_} := \mathbb{E}_{\{\}\to\{p_i,x_i\}}[1, 0]$$

In[∘]:=
$$m_{i\_,j\_\to k\_} := \mathbb{E}_{\{\pi_i,\xi_i,\pi_j,\xi_j\}\to\{p_k,x_k\}}\left[1, -\xi_i\,\pi_j + (\pi_i+\pi_j)\,p_k + (\xi_i+\xi_j)\,x_k\right]$$

Reidemeister 3

In[∘]:= $R_{1,2}\,R_{4,3}\,R_{5,6}$

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,p_2,p_3,p_4,p_5,p_6,x_1,x_2,x_3,x_4,x_5,x_6\}}\left[T^{3/2}, (-1+T)\,p_1\,x_2 + (1-T)\,p_2\,x_2 + (1-T)\,p_3\,x_3 + (-1+T)\,p_4\,x_3 + (-1+T)\,p_5\,x_6 + (1-T)\,p_6\,x_6\right]$

In[∘]:= $R_{1,2}\,R_{4,3}\,R_{5,6} \,//\, m_{1,4\to 1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,p_2,p_3,p_5,p_6,x_1,x_2,x_3,x_5,x_6\}}\left[T^{3/2}, -p_1\,x_2 + T\,p_1\,x_2 + p_2\,x_2 - T\,p_2\,x_2 - p_1\,x_3 + T\,p_1\,x_3 + p_3\,x_3 - T\,p_3\,x_3 - p_5\,x_6 + T\,p_5\,x_6 + p_6\,x_6 - T\,p_6\,x_6\right]$

In[∘]:= $(R_{1,2}\,R_{4,3}\,R_{5,6} \,//\, m_{1,4\to 1}\,m_{2,5\to 2}\,m_{3,6\to 3}) \equiv (R_{2,3}\,R_{1,6}\,R_{4,5} \,//\, m_{1,4\to 1}\,m_{2,5\to 2}\,m_{3,6\to 3})$

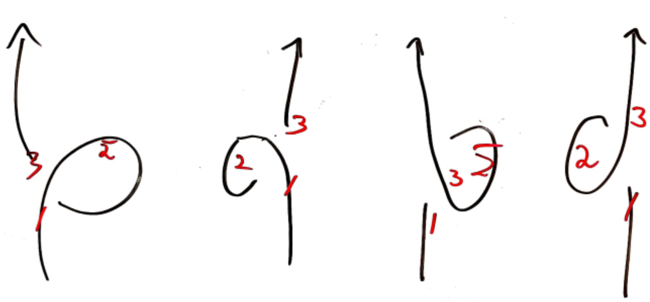
Out[∘]= True

Reidemeister 2

In[∘]:= $\left\{\left(\overline{R}_{1,2}\,R_{3,4} \,//\, m_{1,3\to 1}\,m_{2,4\to 2}\right) \equiv \eta_1\,\eta_2,\right.$
$\left.\left(R_{1,4}\,\overline{R}_{5,2}\,\overline{C}_3 \,//\, m_{2,4\to 2} \,//\, m_{1,3\to 1} \,//\, m_{1,5\to 1}\right) \equiv \overline{C}_1\,\eta_2\right\}$

Out[∘]= {True, True}

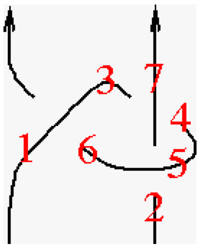
Reidemeister 1's



In[∘]:= $\{(\overline{C}_2\, R_{1,3} \,//\, m_{1,2\to 1} \,//\, m_{1,3\to 1}) \equiv \eta_1,\ (\overline{C}_2\, \overline{R}_{3,1} \,//\, m_{1,2\to 1} \,//\, m_{1,3\to 1}) \equiv \eta_1,\ (C_2\, \overline{R}_{1,3} \,//\, m_{1,2\to 1} \,//\, m_{1,3\to 1}) \equiv \eta_1,\ (C_2\, R_{3,1} \,//\, m_{1,2\to 1} \,//\, m_{1,3\to 1}) \equiv \eta_1\}$

Out[∘]= {True, True, True, True}

The “First Tangle”



In[∘]:= Factor /@ $(z = R_{1,6}\, \overline{C}_3\, \overline{R}_{7,4}\, \overline{R}_{5,2} \,//\, m_{1,3\to 1} \,//\, m_{1,4\to 1} \,//\, m_{1,5\to 1} \,//\, m_{1,6\to 1} \,//\, m_{2,7\to 2})$

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}\left[\frac{T}{-1+2T}, \frac{(-1+T)\,(p_1-p_2)\,(T\,x_1-x_2)}{-1+2T}\right]$

In[∘]:= ZF[Knot[4, 1]]

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{T}{1-3T+T^2}, 0\right]$

In[∘]:= Alexander[Knot[4, 1]][T]

Out[∘]= $3-\frac{1}{T}-T$

In[∘]:= Timing@Table[Simplify[Alexander[K][T] ZF[K][[1]]], {K, AllKnots[{3, 9}]}]

Out[∘]= {26.1719, {1, -1, 1, 1, -1, -1, 1, 1, 1, 1, 1, 1, -1, 1, -1, -1, -1, -1, -1, -1, 1, 1, -1, 1, -1, 1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, 1, 1, 1, 1, 1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, 1, 1, 1, -1, -1, 1, 1, -1, -1, 1, -1, 1, 1, -1, 1, 1, -1, -1, 1, -1, 1, 1, -1, 1, 1, -1, -1, 1, -1, -1, 1, -1, 1}}

In[∘]:= Timing@ZF[GST48]

Out[∘]= $\{10.2969, \mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{T^8}{1-2T+T^2+2T^4-5T^5+2T^6+7T^7-13T^8+7T^9+2T^{10}-5T^{11}+2T^{12}+T^{14}-2T^{15}+T^{16}}, 0\right]\}$