

In[]:= **Once** [<< KnotTheory`]

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at <http://katlas.org/wiki/KnotTheory>.

In[]:= $\chi[\text{cond}_] := \text{If}[\text{TrueQ}[\text{cond}], 1, 0];$

In[]:= $\mathcal{E} // \text{m}[\mathbf{i}_, \mathbf{j}_ \rightarrow \mathbf{k}_] :=$
 $\text{Expand}[\mathcal{E} /. \mathbf{e}_{r,s,t}[\mathbf{i}] \mathbf{e}_{u,v,w}[\mathbf{j}] \rightarrow \chi[t == v] (-1)^u (s+t) \mathbf{e}_{(r+u) \bmod 2, s, w}[\mathbf{k}]]$

In[]:= **KBasis** [{ **i** }] :=
{ **e**_{0,0,0} [**i**], **e**_{0,0,1} [**i**], **e**_{0,1,0} [**i**], **e**_{0,1,1} [**i**], **e**_{1,0,0} [**i**], **e**_{1,0,1} [**i**], **e**_{1,1,0} [**i**], **e**_{1,1,1} [**i**]
KBasis [{ **i**_, **is**_ }] := **Flatten** @ **Outer** [**Times**, **KBasis** [{ **i** }], **KBasis** [{ **is** }]]

In[]:= $\eta[\mathbf{i}_] := \mathbf{e}_{0,0,0}[\mathbf{i}] + \mathbf{e}_{0,1,1}[\mathbf{i}]$

In[]:= **lhs** = $\eta[1] \times \text{KBasis}[\{2\}] // \text{Expand} // \text{m}[1, 2 \rightarrow 1];$ **lhs** == **KBasis** [{ 1 }]

Out[]:= True

In[]:= **lhs** = $\eta[2] \times \text{KBasis}[\{1\}] // \text{Expand} // \text{m}[1, 2 \rightarrow 1];$ **lhs** == **KBasis** [{ 1 }]

Out[]:= True

In[]:= **lhs** = **KBasis** [{ 1, 2, 3 }] // **m**[1, 2 → 1] // **m**[1, 3 → 1];
rhs = **KBasis** [{ 1, 2, 3 }] // **m**[2, 3 → 2] // **m**[1, 2 → 1]; **lhs** == **rhs**

Out[]:= True

In[]:= $\mathbf{R}[\mathbf{i}_, \mathbf{j}_] := \mathbf{a}1 \mathbf{e}_{0,0,0}[\mathbf{i}] \mathbf{e}_{0,0,0}[\mathbf{j}] + \mathbf{a}2 \mathbf{e}_{0,1,1}[\mathbf{i}] \mathbf{e}_{0,0,0}[\mathbf{j}] +$
 $\mathbf{a}3 \mathbf{T} \mathbf{e}_{0,0,0}[\mathbf{i}] \mathbf{e}_{0,1,1}[\mathbf{j}] - \mathbf{a}4 (1 - \mathbf{T}) \mathbf{e}_{1,1,0}[\mathbf{i}] \mathbf{e}_{1,0,1}[\mathbf{j}] - \mathbf{a}5 \mathbf{T} \mathbf{e}_{0,1,1}[\mathbf{i}] \mathbf{e}_{0,1,1}[\mathbf{j}]$

In[]:= **lhs** = **R**[1, 2] × **R**[4, 3] × **R**[5, 6] // **Expand** // **m**[1, 4 → 1] // **m**[2, 5 → 2] // **m**[3, 6 → 3];
rhs = **R**[2, 3] × **R**[1, 4] × **R**[5, 6] // **Expand** // **m**[1, 5 → 1] // **m**[2, 6 → 2] // **m**[3, 4 → 3];
diff = **lhs** - **rhs**

Out[]:= $\mathbf{a}1^2 \mathbf{a}4 \mathbf{e}_{0,0,0}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1] - \mathbf{a}1 \mathbf{a}4^2 \mathbf{e}_{0,0,0}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1] -$
 $\mathbf{a}1^2 \mathbf{a}4 \mathbf{T} \mathbf{e}_{0,0,0}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1] - \mathbf{a}2 \mathbf{a}3 \mathbf{a}4 \mathbf{T} \mathbf{e}_{0,0,0}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1] +$
 $2 \mathbf{a}1 \mathbf{a}4^2 \mathbf{T} \mathbf{e}_{0,0,0}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1] + \mathbf{a}2 \mathbf{a}3 \mathbf{a}4 \mathbf{T}^2 \mathbf{e}_{0,0,0}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1] -$
 $\mathbf{a}1 \mathbf{a}4^2 \mathbf{T}^2 \mathbf{e}_{0,0,0}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1] + \mathbf{a}2 \mathbf{a}3 \mathbf{a}4 \mathbf{T} \mathbf{e}_{0,1,1}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1] -$
 $\mathbf{a}4^2 \mathbf{a}5 \mathbf{T} \mathbf{e}_{0,1,1}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1] - \mathbf{a}2 \mathbf{a}3 \mathbf{a}4 \mathbf{T}^2 \mathbf{e}_{0,1,1}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1] +$
 $2 \mathbf{a}4^2 \mathbf{a}5 \mathbf{T}^2 \mathbf{e}_{0,1,1}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1] - \mathbf{a}4 \mathbf{a}5^2 \mathbf{T}^2 \mathbf{e}_{0,1,1}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1] -$
 $\mathbf{a}4^2 \mathbf{a}5 \mathbf{T}^3 \mathbf{e}_{0,1,1}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1] + \mathbf{a}4 \mathbf{a}5^2 \mathbf{T}^3 \mathbf{e}_{0,1,1}[2] \mathbf{e}_{1,0,1}[3] \mathbf{e}_{1,1,0}[1]$

In[]:= **Coefficient** [**diff**, **e**_{0,0,0} [2] **e**_{1,0,1} [3] **e**_{1,1,0} [1]]

Out[]:= $\mathbf{a}1^2 \mathbf{a}4 - \mathbf{a}1 \mathbf{a}4^2 - \mathbf{a}1^2 \mathbf{a}4 \mathbf{T} - \mathbf{a}2 \mathbf{a}3 \mathbf{a}4 \mathbf{T} + 2 \mathbf{a}1 \mathbf{a}4^2 \mathbf{T} + \mathbf{a}2 \mathbf{a}3 \mathbf{a}4 \mathbf{T}^2 - \mathbf{a}1 \mathbf{a}4^2 \mathbf{T}^2$

In[*]:= **Block** [{**a1** = **1**}, **Coefficient**[**diff**, **e**_{0,0,0}[**2**] **e**_{1,0,1}[**3**] **e**_{1,1,0}[**1**]]]

Out[*]:= $a4 - a4^2 - a4 T - a2 a3 a4 T + 2 a4^2 T + a2 a3 a4 T^2 - a4^2 T^2$

In[*]:= **Block** [{**a1** = **1**, **a4** = **1**}, **Coefficient**[**diff**, **e**_{0,0,0}[**2**] **e**_{1,0,1}[**3**] **e**_{1,1,0}[**1**]]]

Out[*]:= $T - a2 a3 T - T^2 + a2 a3 T^2$

In[*]:= **Block** [{**a1** = **1**, **a4** = **1**, **a3** = **a2**}, **Coefficient**[**diff**, **e**_{0,0,0}[**2**] **e**_{1,0,1}[**3**] **e**_{1,1,0}[**1**]]]

Out[*]:= $T - a2^2 T - T^2 + a2^2 T^2$

In[*]:= **Block** [{**a1** = **1**, **a4** = **1**, **a3** = **a2**}, **Coefficient**[**diff**, **e**_{0,1,1}[**2**] **e**_{1,0,1}[**3**] **e**_{1,1,0}[**1**]]]

Out[*]:= $a2^2 T - a5 T - a2^2 T^2 + 2 a5 T^2 - a5^2 T^2 - a5 T^3 + a5^2 T^3$

In[*]:= **Block** [{**a1** = **1**, **a4** = **1**, **a3** = **a2**, **a5** = **1**}, **Coefficient**[**diff**, **e**_{0,1,1}[**2**] **e**_{1,0,1}[**3**] **e**_{1,1,0}[**1**]]]

Out[*]:= $-T + a2^2 T + T^2 - a2^2 T^2$

In[*]:=
$$\bar{R}[i_, j_] := b1 e_{0,0,0}[i] e_{0,0,0}[j] + b2 e_{0,1,1}[i] e_{0,0,0}[j] + b3 T^{-1} e_{0,0,0}[i] e_{0,1,1}[j] - b4 (1 - T^{-1}) e_{1,1,0}[i] e_{1,0,1}[j] - b5 T^{-1} e_{0,1,1}[i] e_{0,1,1}[j]$$

In[*]:= **lhs** = **R**[**1**, **2**] \bar{R} [**3**, **4**] // **Expand** // **m**[**1**, **3** → **1**] // **m**[**2**, **4** → **2**];

rhs = η [**1**] × η [**2**] // **Expand**;

diff = **lhs** - **rhs**;

Block [{**a1** = **1**, **a4** = **1**, **a3** = **a2**, **a5** = **1**}, **diff**]

Out[*]:= $-e_{0,0,0}[1] e_{0,0,0}[2] + b1 e_{0,0,0}[1] e_{0,0,0}[2] - e_{0,0,0}[2] e_{0,1,1}[1] + a2 b2 e_{0,0,0}[2] e_{0,1,1}[1] - e_{0,0,0}[1] e_{0,1,1}[2] + a2 b3 e_{0,0,0}[1] e_{0,1,1}[2] - e_{0,1,1}[1] e_{0,1,1}[2] + b5 e_{0,1,1}[1] e_{0,1,1}[2] + b3 e_{1,0,1}[2] e_{1,1,0}[1] - a2 b4 e_{1,0,1}[2] e_{1,1,0}[1] - \frac{b3 e_{1,0,1}[2] e_{1,1,0}[1]}{T} + \frac{a2 b4 e_{1,0,1}[2] e_{1,1,0}[1]}{T}$

In[*]:= **Block** [{**a1** = **1**, **a4** = **1**, **a3** = **a2**, **a5** = **1**}, **Coefficient**[**diff**, **e**_{0,0,0}[**1**] **e**_{0,0,0}[**2**]]]

Out[*]:= $-1 + b1$

In[*]:= **Block** [{**a1** = **1**, **a4** = **1**, **a3** = **a2**, **a5** = **1**, **b1** = **1**}, **diff**]

Out[*]:= $-e_{0,0,0}[2] e_{0,1,1}[1] + a2 b2 e_{0,0,0}[2] e_{0,1,1}[1] - e_{0,0,0}[1] e_{0,1,1}[2] + a2 b3 e_{0,0,0}[1] e_{0,1,1}[2] - e_{0,1,1}[1] e_{0,1,1}[2] + b5 e_{0,1,1}[1] e_{0,1,1}[2] + b3 e_{1,0,1}[2] e_{1,1,0}[1] - a2 b4 e_{1,0,1}[2] e_{1,1,0}[1] - \frac{b3 e_{1,0,1}[2] e_{1,1,0}[1]}{T} + \frac{a2 b4 e_{1,0,1}[2] e_{1,1,0}[1]}{T}$

In[*]:= **Block** [{**a1** = **1**, **a4** = **1**, **a3** = **a2**, **a5** = **1**, **b1** = **1**}, **Coefficient**[**diff**, **e**_{0,1,1}[**1**] **e**_{0,1,1}[**2**]]]

Out[*]:= $-1 + b5$

In[*]:= **Block** [{**a1** = **1**, **a4** = **1**, **a3** = **a2**, **a5** = **1**, **b1** = **1**, **b5** = **1**}, **diff**]

Out[*]:= $-e_{0,0,0}[2] e_{0,1,1}[1] + a2 b2 e_{0,0,0}[2] e_{0,1,1}[1] - e_{0,0,0}[1] e_{0,1,1}[2] + a2 b3 e_{0,0,0}[1] e_{0,1,1}[2] + b3 e_{1,0,1}[2] e_{1,1,0}[1] - a2 b4 e_{1,0,1}[2] e_{1,1,0}[1] - \frac{b3 e_{1,0,1}[2] e_{1,1,0}[1]}{T} + \frac{a2 b4 e_{1,0,1}[2] e_{1,1,0}[1]}{T}$

In[*]:= **Block**[{**a1 = 1, a4 = 1, a3 = a2, a5 = 1, b1 = 1, b5 = 1**}, **Coefficient**[**diff, e_{0,0,0}[2] e_{0,1,1}[1]**]]

Out[*]:= $-1 + a_2 b_2$

In[*]:= **Block**[{**a1 = 1, a4 = 1, a3 = a2, a5 = 1, b1 = 1, b5 = 1, b2 = a2**}, **diff**] /. **a2² → 1**

Out[*]:= $-e_{0,0,0}[1] e_{0,1,1}[2] + a_2 b_3 e_{0,0,0}[1] e_{0,1,1}[2] + b_3 e_{1,0,1}[2] e_{1,1,0}[1] -$
 $a_2 b_4 e_{1,0,1}[2] e_{1,1,0}[1] - \frac{b_3 e_{1,0,1}[2] e_{1,1,0}[1]}{\tau} + \frac{a_2 b_4 e_{1,0,1}[2] e_{1,1,0}[1]}{\tau}$

In[*]:= **Block**[{**a1 = 1, a4 = 1, a3 = a2, a5 = 1, b1 = 1, b5 = 1, b2 = a2**},
Coefficient[**diff, e_{0,0,0}[1] e_{0,1,1}[2]**]] /. **a2² → 1**

Out[*]:= $-1 + a_2 b_3$

In[*]:= **Block**[{**a1 = 1, a4 = 1, a3 = a2, a5 = 1, b1 = 1, b5 = 1, b2 = a2, b3 = a2**}, **diff**] /. **a2² → 1**

Out[*]:= $a_2 e_{1,0,1}[2] e_{1,1,0}[1] - a_2 b_4 e_{1,0,1}[2] e_{1,1,0}[1] - \frac{a_2 e_{1,0,1}[2] e_{1,1,0}[1]}{\tau} + \frac{a_2 b_4 e_{1,0,1}[2] e_{1,1,0}[1]}{\tau}$

In[*]:= **Block**[{**a1 = 1, a4 = 1, a3 = a2, a5 = 1, b1 = 1, b5 = 1, b2 = a2, b3 = a2, b4 = 1**}, **diff**] /.
a2² → 1

Out[*]:= 0

In[*]:= **lhs = R**[1, 2] **R̄**[3, 4] // **Expand** // **m**[1, 3 → 1] // **m**[4, 2 → 2]

rhs = η[1] × **η**[2] // **Expand**

Simplify[**diff = lhs - rhs**]

Out[*]:= $a_1 b_1 e_{0,0,0}[1] e_{0,0,0}[2] + a_2 b_2 e_{0,0,0}[2] e_{0,1,1}[1] +$
 $a_3 b_3 e_{0,0,0}[1] e_{0,1,1}[2] + a_5 b_5 e_{0,1,1}[1] e_{0,1,1}[2] - a_4 b_1 e_{1,0,1}[2] e_{1,1,0}[1] -$
 $a_5 b_4 e_{1,0,1}[2] e_{1,1,0}[1] + a_4 b_1 \tau e_{1,0,1}[2] e_{1,1,0}[1] + a_5 b_4 \tau e_{1,0,1}[2] e_{1,1,0}[1]$

Out[*]:= $e_{0,0,0}[1] e_{0,0,0}[2] + e_{0,0,0}[2] e_{0,1,1}[1] + e_{0,0,0}[1] e_{0,1,1}[2] + e_{0,1,1}[1] e_{0,1,1}[2]$

Out[*]:= $(-1 + a_2 b_2) e_{0,0,0}[2] e_{0,1,1}[1] - e_{0,1,1}[1] e_{0,1,1}[2] + a_5 b_5 e_{0,1,1}[1] e_{0,1,1}[2] +$
 $e_{0,0,0}[1] ((-1 + a_1 b_1) e_{0,0,0}[2] + (-1 + a_3 b_3) e_{0,1,1}[2]) - a_4 b_1 e_{1,0,1}[2] e_{1,1,0}[1] -$
 $a_5 b_4 e_{1,0,1}[2] e_{1,1,0}[1] + a_4 b_1 \tau e_{1,0,1}[2] e_{1,1,0}[1] + a_5 b_4 \tau e_{1,0,1}[2] e_{1,1,0}[1]$

In[*]:= **Block**[{**a1 = 1, a4 = 1, a3 = a2, a5 = 1, b1 = 1, b5 = 1, b2 = a2, b3 = a2, b4 = 1**}, **diff**] //
Simplify

Out[*]:= $(-1 + a_2^2) e_{0,0,0}[2] e_{0,1,1}[1] + (-1 + a_2^2) e_{0,0,0}[1] e_{0,1,1}[2] + 2 \times (-1 + \tau) e_{1,0,1}[2] e_{1,1,0}[1]$