Homework Assignment 6

Over the base field $\mathbb{Q}$, recall that sl_{2+}^{ϵ} is the Lie algebra with generators $\{\mathbf{y}, \mathbf{b}, \mathbf{a}, \mathbf{x}\}$ and with brackets

$$[\mathbf{a}, \mathbf{x}] = \mathbf{x}, \quad [\mathbf{b}, \mathbf{y}] = -\epsilon\mathbf{y}, \quad [\mathbf{a}, \mathbf{b}] = 0, \quad [\mathbf{a}, \mathbf{y}] = -\mathbf{y},$$

$$[\mathbf{b}, \mathbf{x}] = \epsilon\mathbf{x}, \quad [\mathbf{x}, \mathbf{y}] = \mathbf{b} + \epsilon\mathbf{a}.$$

Recall also that sl_2 is the Lie algebra of 2×2 matrices whose trace is 0 and with $[A, B] \coloneqq AB - BA$. Note that if $\mathfrak{a}$ and $\mathfrak{b}$ are Lie algebras, then so is their direct sum $\mathfrak{a} \oplus \mathfrak{b}$, taken with the obvious bracket. Finally, a 1-dimensional vector space $\langle t \rangle$ is a Lie algebra (in only one way), with $[t, t] = 0$.

Problem 1. Show that for $\epsilon \neq 0$ the Lie algebra sl_{2+}^{ϵ} is isomorphic to the direct sum $sl_2 \oplus \langle t \rangle$. (Hint. What properties would the image of t in sl_{2+}^{ϵ} need to have?)

Problem 2. Show that for $\epsilon = 0$ the Lie algebra sl_{2+}^{0} is not a direct sum of smaller algebras.

(That's kind of unexpected, isn't it? You'd think that on a one-parameter path of Lie algebras some miracle could occur, and some Lie algebra in the middle of the path might have a decomposition into smaller ones even if the generic case is indecomposable. But what we see here is the opposite!)