Definition 1. Let sl_{2+}^{ϵ} be the Lie algebra with generators $\{\mathbf{y},\mathbf{b},\mathbf{a},\mathbf{x}\}$ and with commutation relations

$$[\mathbf{a},\mathbf{x}]=\mathbf{x},\quad [\mathbf{b},\mathbf{y}]=-\epsilon\mathbf{y},\quad [\mathbf{a},\mathbf{b}]=0,\quad [\mathbf{a},\mathbf{y}]=-\mathbf{y},$$

$$[\mathbf{b},\mathbf{x}]=\epsilon\mathbf{x},\quad [\mathbf{x},\mathbf{y}]=\mathbf{b}+\epsilon\mathbf{a}.$$

Definition 2. Let $CU\coloneqq\mathcal{U}(sl_{2+}^{\epsilon})$ be the universal enveloping algebra of sl_{2+}^{ϵ}. Namely, CU is the associative algebra $A\langle\mathbf{y},\mathbf{b},\mathbf{a},\mathbf{x}\rangle$ generated by the same $\{\mathbf{y},\mathbf{b},\mathbf{a},\mathbf{x}\}$, subject to the same relations as above. We denote the multiplication map of CU by $cm_k^{ij}\colon CU_i\otimes CU_j\to CU_k$. CU is a Hopf algebra in the standard way; namely, with its given associative algebra structure and with unit $c\eta\colon\mathbb{Q}\to CU$, counit $c\varepsilon\colon CU\to\mathbb{Q}$, antipode $cS\colon CU\to CU$, and coproduct $c\Delta\colon CU\to CU\otimes CU$ given as follows:

$$\begin{aligned}c\eta_i(\lambda)&=\lambda\cdot 1_i,\\ c\varepsilon^i(1_i,y_i,b_i,a_i,x_i)&=(1,0,0,0,0),\\ cS_i(y_i,b_i,a_i,x_i)&=(-y_i,-b_i,-a_i,-x_i),\\ c\Delta^i_{jk}(y_i,b_i,a_i,x_i)&=(y_j+y_k,b_j+b_k,a_j+a_k,x_j+x_k).\end{aligned}$$

Definition 3. Let QU, a "quantization" of CU, be the associative algebra $A\langle\mathbf{y},\mathbf{b},\mathbf{a},\mathbf{x}\rangle[\![\hbar]\!]$ over the ring $\mathbb{Q}[\![\hbar]\!]$ modulo to the relations

$$[\mathbf{a},\mathbf{x}]=\mathbf{x},\quad [\mathbf{b},\mathbf{y}]=-\epsilon\mathbf{y},\quad [\mathbf{a},\mathbf{b}]=0,\quad [\mathbf{a},\mathbf{y}]=-\mathbf{y},$$

$$[\mathbf{b},\mathbf{x}]=\epsilon\mathbf{x},\quad \mathbf{xy}-q\mathbf{yx}=\frac{1-\mathbf{AB}}{\hbar},$$

where $q\coloneqq\mathbb{e}^{\hbar\epsilon}$, $\mathbf{A}\coloneqq\mathbb{e}^{-\hbar\epsilon\mathbf{a}}$, and $\mathbf{B}\coloneqq\mathbb{e}^{-\hbar\mathbf{b}}$. We denote the multiplication map of QU with $qm_k^{ij}\colon QU_i\otimes QU_j\to QU_k$.

We also set

$$\begin{aligned}q\eta_i(\lambda)&=\lambda\cdot 1_i,\\ q\varepsilon^i(1_i,\mathbf{y}_i,\mathbf{b}_i,\mathbf{a}_i,\mathbf{x}_i)&=(1,0,0,0,0),\\ qS_i(\mathbf{y}_i,\mathbf{b}_i,\mathbf{a}_i,\mathbf{x}_i)&=(-\mathbf{B}_i^{-1}\mathbf{y}_i,-\mathbf{b}_i,-\mathbf{a}_i,-\mathbf{A}_i^{-1}\mathbf{x}_i),\\ q\Delta^i_{jk}(\mathbf{y}_i,\mathbf{b}_i,\mathbf{a}_i,\mathbf{x}_i)&=(\mathbf{y}_j+\mathbf{B}_j\mathbf{y}_k,\mathbf{b}_j+\mathbf{b}_k,\mathbf{a}_j+\mathbf{a}_k,\mathbf{x}_j+\mathbf{A}_j\mathbf{x}_k).\end{aligned}$$

Definition 4. Let R_{ij} be the element of $QU_i\otimes QU_j$ given by the following formula:

$$R_{ij}=\sum_{m,n\geq 0}\frac{\mathbf{y}_i^n\mathbf{b}_i^m(\hbar\mathbf{a}_j)^m(\hbar\mathbf{x}_j)^n}{m![n]_q!}\sim\mathbb{e}^{\hbar\mathbf{b}_i\mathbf{a}_j}\mathbb{e}_q^{\hbar\mathbf{y}_i\mathbf{x}_j},$$

where
$$[n]_q!\coloneqq[1]_q[2]_q\cdots[n]_q,$$

$$[k]_q\coloneqq 1+q+q^2+\ldots+q^{k-1}=\frac{q^k-1}{q-1},$$

and
$$\mathbb{e}_q^z\coloneqq\sum_{n=0}^{\infty}\frac{z^n}{[n]_q!}.$$

Convention 5. For both CU and QU, we always order the generators in the order **ybax**.

Questions.
1. Why call sl_{2+}^{ϵ} that way?
2. How do we compute in CU?
3. How did QU come about? How is it a "quatization" of CU? How is (related to) a "quantization" of sl_2?
4. That funny formula for R, wherefore?
5. How do we verify all the Hopf+R properties for QU?
6. How do we compute in QU?
7. Is this **it**? A poly-time strong tangle invariant, homomorphic for m, Δ, S?

Theorem 6. *(PBW, Poincaré-Birkhoff-Witt) For any Lie algebra $\mathfrak{g}$ with any choice of ordered basis $\mathbf{X}=(\mathbf{x}_1,\mathbf{x}_2,\ldots)$ with commutative counterpart $X=(x_1,x_2,\ldots)$, the ordering map $\mathbb{O}_X\colon\mathbb{Q}[X]\to\mathcal{U}(\mathfrak{g})$ defined by $x_1^{n_1}x_2^{n_2}\cdot\ldots\mapsto\mathbf{x}_1^{n_1}\mathbf{x}_2^{n_2}\cdot\ldots$ is an isomorphism of vector spaces.*

Proposition 7. $c\eta_i=\mathbb{E}_{\emptyset\to\{i\}}[0]$, $c\varepsilon^i=\mathbb{E}_{\{i\}\to\emptyset}[0]$, $cS_i=\mathbb{E}_{\{i\}\to\{i\}}[-\eta_iy_i-\beta_ib_i-\alpha_ia_i-\xi_ix_i]$, $c\Delta^i_{jk}=\mathbb{E}_{\{i\}\to\{j,k\}}[\eta_i(y_j+y_k)+\beta_i(b_j+b_k)+\alpha_i(a_j+a_k)+\xi_i(x_j+x_k)]$, and $cm_k^{ij}=\mathbb{E}_{\{i\}\to\{j,k\}}[\Lambda]$, where

$$\Lambda=\left(\eta_i+\frac{e^{-\alpha_i-\epsilon\beta_i}\eta_j}{1+\epsilon\eta_j\xi_i}\right)y_k+\left(\beta_i+\beta_j+\frac{\log\left(1+\epsilon\eta_j\xi_i\right)}{\epsilon}\right)b_k$$
$$+\left(\alpha_i+\alpha_j+\log\left(1+\epsilon\eta_j\xi_i\right)\right)a_k+\left(\frac{e^{-\alpha_j-\epsilon\beta_j}\xi_i}{1+\epsilon\eta_j\xi_i}+\xi_j\right)x_k.$$

$\frac{1}{2}$**Proof.** We show that

$$\mathbb{e}^{\eta_iy_i+\beta_ib_i+\alpha_ia_i+\xi_ix_i+\eta_jy_j+\beta_jb_j+\alpha_ja_j+\xi_jx_j}/\!\!/\mathbb{O}_{i,j}/\!\!/cm_k^{ij}=\mathbb{e}^{\Lambda}/\!\!/\mathbb{O}_k$$

by computing in a faithful 2D representation ρ of CU:

```
{ŷ = (0 0; ϵ 0), b̂ = (0 0; 0 -ϵ), â = (1 0; 0 0), x̂ = (0 1; 0 0)};
{â.x̂ - x̂.â == x̂, â.ŷ - ŷ.â == -ŷ, b̂.ŷ - ŷ.b̂ == -ϵ ŷ,
 b̂.x̂ - x̂.b̂ == ϵ x̂, x̂.ŷ - ŷ.x̂ == b̂ + ϵ â}
```

```
{True, True, True, True, True}
```

```
Simplify@With[{𝔼 = MatrixExp},
  𝔼[ηi ŷ].𝔼[βi b̂].𝔼[αi â].𝔼[ξi x̂].𝔼[ηj ŷ].𝔼[βj b̂].
   𝔼[αj â].𝔼[ξj x̂] == 𝔼[ŷ ∂yk Λ].𝔼[b̂ ∂bk Λ].𝔼[â ∂ak Λ].
   𝔼[x̂ ∂xk Λ]]
```

```
True
```

```
Series[Λ, {ϵ, 0, 2}]
```

$$\begin{aligned}&\left(a_k(\alpha_i+\alpha_j)+y_k\left(\eta_i+\mathbb{e}^{-\alpha_i}\eta_j\right)+b_k(\beta_i+\beta_j+\eta_j\xi_i)+x_k\left(\mathbb{e}^{-\alpha_j}\xi_i+\xi_j\right)\right)+\\ &\left(a_k\eta_j\xi_i-\frac{1}{2}b_k\eta_j^2\xi_i^2-\mathbb{e}^{-\alpha_i}y_k\eta_j(\beta_i+\eta_j\xi_i)-\mathbb{e}^{-\alpha_j}x_k\xi_i(\beta_j+\eta_j\xi_i)\right)\epsilon+\\ &\left(-\frac{1}{2}a_k\eta_j^2\xi_i^2+\frac{1}{3}b_k\eta_j^3\xi_i^3+\frac{1}{2}\mathbb{e}^{-\alpha_i}y_k\eta_j\left(\beta_i^2+2\beta_i\eta_j\xi_i+2\eta_j^2\xi_i^2\right)+\frac{1}{2}\mathbb{e}^{-\alpha_j}x_k\xi_i\left(\beta_j^2+2\beta_j\eta_j\xi_i+2\eta_j^2\xi_i^2\right)\right)\epsilon^2+O[\epsilon]^3\end{aligned}$$