May 28 Scratch

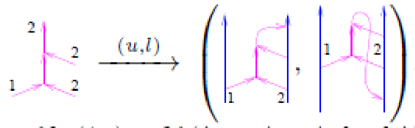
May-28-12
6:00 PM

Work required.

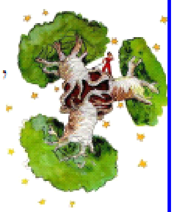
From Talks/Montpelier-1006:

Wheels and Trees. With $\mathcal{P}$ for Primitives,

$$0 \longrightarrow \langle\text{wheels}\rangle \xrightarrow{\iota} \mathcal{PA}^w(\uparrow_n) \underset{l}{\overset{u}{\rightleftarrows}} \langle\text{trees}\rangle \longrightarrow 0,$$

with 

So $\operatorname{proj}\mathcal{K}^w(\uparrow_n) \cong \mathcal{U}(\langle\text{trees}\rangle \ltimes \langle\text{wheels}\rangle)$.



trees atop a wheel, and a little prince

Some A-T Notions. $\mathfrak{a}_n$ is the vector space with basis $x_1, \dots, x_n$, $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ is the free Lie algebra, $\text{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ is the free associative algera "of words", $\text{tr}: \text{Ass}_n^+ \to \mathfrak{tr}_n = \text{Ass}_n^+/(x_{i_1}x_{i_2}\cdots x_{i_m} = x_{i_2}\cdots x_{i_m}x_{i_1})$ is the "trace" into "cyclic words", $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ are all the derivations, and

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n : \forall i\ \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

are "tangential derivations", so $D \leftrightarrow (a_1, \dots, a_n)$ is a vector space isomorphism $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_n \mathfrak{lie}_n$. Finally, $\text{div}: \mathfrak{tder}_n \to \mathfrak{tr}_n$ is $(a_1, \dots, a_n) \mapsto \sum_k \text{tr}(x_k(\partial_k a_k))$, where for $a \in \text{Ass}_n^+$, $\partial_k a \in \text{Ass}_n$ is determined by $a = \sum_k (\partial_k a) x_k$, and $j: \text{TAut}_n = \exp(\mathfrak{tder}_n) \to \mathfrak{tr}_n$ is $j(e^D) = \frac{e^D - 1}{D} \cdot \text{div}\, D$.

Theorem. Everything matches. $\langle\text{trees}\rangle$ is $\mathfrak{a}_n \oplus \mathfrak{tder}_n$ as Lie algebras, $\langle\text{wheels}\rangle$ is $\mathfrak{tr}_n$ as $\langle\text{trees}\rangle$ / $\mathfrak{tder}_n$-modules, $\text{div}\, D = \iota^{-1}(u - l)(D)$, and $e^{uD}e^{-lD} = e^{jD}$.

Differential Operators. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\text{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\text{ad}\, x$: $(x\varphi)(y) := \varphi([x, y])$.

Trees become vector fields and $uD \mapsto lD$ is $D \mapsto D^*$. So $\text{div}\, D$ is $D - D^*$ and $jD = \log(e^D(e^D)^*) = \int_0^1 dt e^{tD}\, \text{div}\, D$.

R3

(1) $V \cdot (\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow\uparrow\uparrow)$

(2) $VV^* = I$ in $\mathcal{A}^w(\uparrow\uparrow)$ (3) $V \cdot \Delta(\omega) = \omega \otimes \omega$ in $\mathcal{A}^w(\uparrow\uparrow)$

Alekseev-Torossian statement. There are elements $F \in \text{TAut}_2$ and $a \in \mathfrak{tr}_1$ such that

$$F(x + y) = \log e^x e^y \quad \text{and} \quad jF = a(x) + a(y) - a(\log e^x e^y).$$

Theorem. The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.

Proof. Write $V = e^c e^{uD}$ with $c \in \mathfrak{tr}_2$, $D \in \mathfrak{tder}_2$, and $\omega = e^b$ with $b \in \mathfrak{tr}_1$. Then (1) $\Leftrightarrow e^{uD}(x + y)e^{-uD} = \log e^x e^y$, (2) $\Leftrightarrow I = e^c e^{uD}(e^{uD})^* e^c = e^{2c}e^{jD}$, and (3) $\Leftrightarrow e^c e^{uD} e^{b(x+y)} = e^{b(x)+b(y)} \Leftrightarrow e^c e^{b(\log e^x e^y)} = e^{b(x)+b(y)} \Leftrightarrow c = b(x) + b(y) - b(\log e^x e^y)$.

$$e^{utD}e^{-ltD} \overset{?}{=} e^{j(tD)}$$

$$\downarrow \tilde{E}_t|_{t=1}$$

$$e^{lD}(uD)e^{-lD} - lD \overset{?}{\underset{0}{=}} e^D(\text{div}\, D)$$

$$e^{lD}(uD - lD)e^{-lD} \overset{?}{=} e^D(\text{div}\, D)$$

$$e^{-ltD}e^{utD} \overset{?}{=} e^{j(tD)}$$

$$\downarrow \tilde{E}_t|_{t=1}$$

$$e^{-uD}(-lD)e^{uD} + uD \overset{?}{=} e^D(\text{div}\, D)$$

Sketch:

1. Review / why "div"?
2. $T: \mathcal{A}^w(\uparrow_n) \xrightarrow{\varphi} \mathcal{U}(I\mathfrak{g})^{\otimes n} \xrightarrow{\text{op}} \mathcal{DO}(\mathfrak{g}^{\oplus}\dots\oplus\mathfrak{g})$
3. T & div.
4. T & R3
5. T & j
6. If time, the three KT equations & their relation with the AT equations.

$$[T(\mapsto) F](x, y) = \sum \varphi_i(x) \ \dots$$

$$\text{Ad}\, e^x(y)$$

$$\left(T(\ell^{1\to 1})F\right)(x,y) = F\left(x, \ell^{\mathrm{ad}x} y\right) = \left(F \circ \mathrm{IAd}\right)(x,y)$$

$$\left(T(R_{12}R_{13}R_{23})F\right)(x,y,z) = \left(F \circ \mathrm{IAd}_{23} \circ \mathrm{IAd}_{13} \circ \mathrm{IAd}_{12}\right)(x,y,z)$$

$$(x,y,z) \longmapsto (x, y^x, z) \longmapsto (x, y^x, z^x) \longmapsto \left(x, y^x, (z^y)^{(y^x)}\right)$$

other side:

$$(x,y,z) \longmapsto (x, y, z^y) \longmapsto (x, y, (z^y)^x) \mapsto \left(x, y^x, (z^y)^x\right)$$