Jan 25 - 2.2.3: Basis Conjugating Automorphisms, 2.3: Finite Type Invariants

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9:02 AM

2.2.3 Basis conjugating Automorphisms of F_n

Artin's Theorem.

$$uB_n \cong \left\{ B \in \mathrm{Aut}^{op}(F_n) : \begin{array}{l} \xi_i /\!/ B = a_i^{-1} \xi_{\beta_i} a_i \\ \prod \xi_i /\!/ B = \prod \xi_i \end{array} \right\}$$

McCool's Theorem

$$wB_n \cong \left\{ B \in \mathrm{Aut}^{op}(F_n) : \xi_i /\!/ B = a^{-1} \xi_{\beta_i} a_i \right\}$$

The conceptual construction:

$$wB_n \xhookrightarrow{\ \iota\ } wB_{n+1} \xhookleftarrow[\text{" inclusion under "}]{\ i_u\ } F_n$$

<u>Claim</u> $i_u(F_n)$ is normalized by $\iota(wB_n)$.

<u>Def</u>

$$\gamma /\!/ B = i_u^{-1}(B^{-1} \gamma B)$$

$$(\xi_i, \xi_{i+1}) /\!/ s_i = (\xi_{i+1}, \xi_i)$$

$$(\xi_i, \xi_{i+1}) /\!/ \sigma_i = (\xi_{i+1}, \xi_{i+1} \xi_i \xi_{i+1}^{-1})$$

$$\xi_j /\!/ \sigma_{ij} = \xi_i \xi_j \xi_i^{-1}$$

Warning 2.8. 1. In ordinary braid theory,

$$PuB_n \xhookrightarrow{\iota} PuB_{n+1} \xleftarrow{i} F_n$$

$PuB_{n+1} \cong PuB_n \ltimes F_n$ Not so in w!

2. Attempting the same w/ "i_0" fails.
3. PvB_n acts on F_n in two ways, yet there are virtual braids in the joint kernel of the two actions.

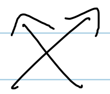
Problem 2.9 Is PwB_n a semi-direct product of free groups?

2.3. F.T.I of v-braids & w-braids.

2.3.1 The "classical/pictorial" approach.

1. F.T for knots — done line

2. semi-virtual xings,

$V($  $) := V($  $) - V($  $)$

"Type m" $\Leftrightarrow V(\text{anything w/} > m \text{ s.v.}) = 0$

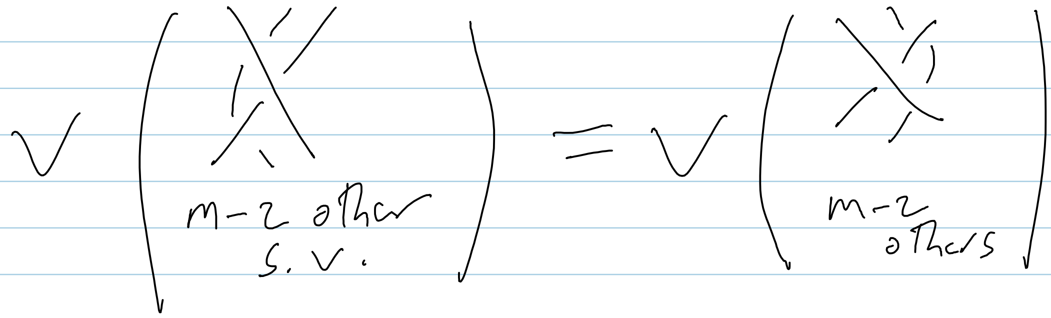
$W = W_m(V) := V\big|_{\text{braids w/ exactly } m \text{ s.v.}}$

$$\mathcal{G}_m \mathcal{D}^v_n := \{m\text{-singular}\} / \{m+1 \text{ singular}\}$$

$=$ arrow diagrams.

Claim. W satisfies the "6T" relation.

Pf.



Now write $\sigma_i = s_i + (\sigma_i - s_i) =$

$$= s_i + \overline{\sigma_i} \ldots$$

Claim. In the w case, W satisfies OC & $\overrightarrow{4T}$. In the u case, ...

Definition. $\mathcal{A}_n^w$, $\mathcal{A}_n^v$, $\mathcal{A}_n^u$

"$\mathcal{A}^{v}_{-}$"

Definition 2.11 An expansion $Z: vB_n \to \mathcal{A}_n^u$

Theorem. An expansion exists iff every W comes from a V.

2.3.2 F.T.I, the "algebraic" approach.

Start from a general group G, define

$$\mathbb{F}G,\ \mathcal{I},\ \mathcal{A}(G),\ \text{expansion}$$

Claim. An invariant of v-braids is

of type m iff it vanishes on $\mathcal{I}^{m+1}$

claim.

$$\mathcal{A}_n^v \twoheadrightarrow \mathcal{A}(vB_n)$$

Def. Expansion $Z: G \to \mathcal{A}(G)$

claim An $\mathcal{A}_n^v$-expansion implies
an expansion & $\mathcal{A}(G) \cong \mathcal{A}_n^v$.

claim. $\mathcal{A}$ is a functor.