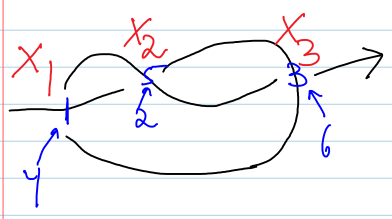
April 4- Section 3.7: The Alexander Polynomial
April-03-12
8:32 AM

* Discussion of the future.

We have $Z^w : K^w(\uparrow) \to \hat{A}^w(\uparrow) = \dfrac{Q[[D_L, D_R, W_i]]}{D_L - D_R = W_1}$

What is it?



	sign of xing	dir of xing $R \to 1, L \to -1$
span $x_1 = (1,4)$	$s_1 = +1$	$d_1 = 1$
span $x_2 = (2,5)$	$s_2 = +1$	$d_2 = -1$
span $x_3 = (3,6)$	$s_3 = +1$	$d_3 = 1$

$$T_{ij} = \left| \begin{array}{c}\text{The lower strand} \\ \text{of } x_j \text{ is in} \\ \text{span } x_i\end{array} \right| \qquad T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A(K)(X) := \det\left(I + T\left(I - X^{-(s_i d_i)}\right)\right)$$

$s_i d_i = \quad 1 \quad -1 \quad 1$

$$A = \begin{vmatrix} 1 & 1-X^{-1} & 0 \\ 1-X & 1 & 0 \\ 1-X & 0 & 1 \end{vmatrix} = 1 - (1-X^{-1})(1-X) = X + X^{-1} - 1$$

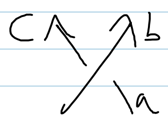
Aside:
$A(K)(1) = 1$
$A(K)(e^x) = 1 + \_\, x^2 + \dots$

Conjecture (or "theorem with unspeakable proofs")

$A(K) = Alex(K)$

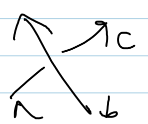
* Verified to 11 xings.
* Follows from later thms.
* Follows from know facts about w-knots (Habiro, Kanenobu, Shima)

Alex: $\det M_{ij}$ where M is given by (up to units)



$\longrightarrow$

	a	b	c
c	-1	1-X	X

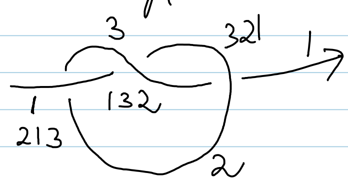


$\longrightarrow$

	a	b	c
c	-X	X-1	1

1. 2. 3.

Example:



$$\begin{array}{c|ccc} 1 & -1 & 1-x & -1 \\ 2 & -1 & x & 1-x \\ 3 & 1-x & -1 & x \end{array} = M \qquad \det M_{11} = x^2+1-x$$

Theorem Let $w: \mathbb{Q}[[x]] \to \hat{\mathcal{A}}^w$ be $x^k \mapsto w_k$.
Then

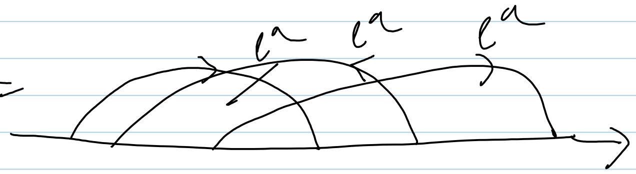
$$Z^w(K) = \exp_{\mathcal{A}^w}(sl_L(K) D_L) \cdot \exp_{\mathcal{A}^w}(sl_R(K) D_R) \quad \} \text{dull}$$

$$\cdot \exp_{\mathcal{A}^w}\left(-w\left(\log_{\mathbb{Q}[[x]]} A(K)(e^x)\right)\right) \quad \} \text{ real stuff}$$

Comment: under $w_k \cdot w_l = w_{k+l}$ $(k, l \geq 2)$, $D_L = D_R = w_1 = 0$
we have

$$Z(K) = A^{-1}(K)(e^x)$$

Proof sketch:

$Z =$  $= ?$

Prelims:

Exercise 3.20 "Commutators commute" (CC) holds in $\mathcal{A}(\uparrow_1)$ (though not in $\mathcal{A}(\uparrow_n)$)

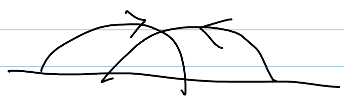
Exercise 3.21. "detached wheels" and "hairy Y's" make sense in $\mathcal{A}(\uparrow)$

Done line

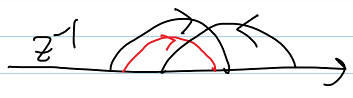
Sketch: 1. The Euler trick

An Euler Interlude. If you know brackets, how do you test exponentials? When's $e^Ae^B = e^Ce^D$?
Bad Idea. Take log and use BCH. You'll want to cry.
Clever Idea. Let E be the Euler derivation, which multiplies each element by its degree (e.g. on $\mathbb{Q}[[\phi]]$, $Ef = \phi\partial_\phi f$, so $Ee^\phi = \phi e^\phi$). Apply $\tilde{E}\zeta := \zeta^{-1}E\zeta$: $\tilde{E}(e^Ae^B) = e^{-B}e^{-A}(e^AAe^B + e^Ae^BB) = e^{-B}Ae^B + B = e^{-\operatorname{ad} B}(A) + B$.

From http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

So apply $\tilde{E}$ to $Z =$  & get

Z^{-1}  $+ Z^{-1}$ 

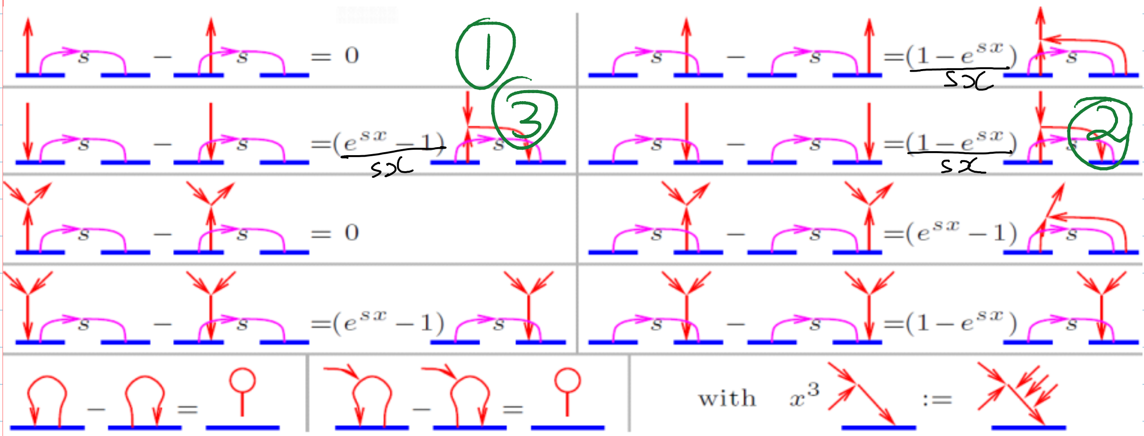
z^{-1}  $+ z^{-1}$ 

On the RHS, get

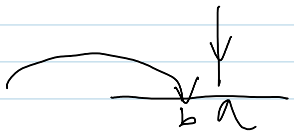
$$(sl_L D_L + sl_R D_R - w[x\,\mathrm{tr}\,((I-B)^{-1}TS\exp(-xS))])$$

with $B := T(\exp(-xS) - I)$.

2.



From http://www.math.toronto.edu/~drorbn/Talks/Toronto-1005/



$e^{\mathrm{ad}\,b}(a) = e^b a e^{-b}$

$\Downarrow$

$e^x = 1 + \frac{e^x - 1}{x} x$

$$e^b a = e^{\mathrm{ad}\,b}(a)e^b = a e^b + \frac{e^{\mathrm{ad}\,b} - 1}{\mathrm{ad}\,b}([b,a]) \cdot e^b$$