April 18 - Section 3.8: Proof of the Alexander Theorem, I

April-16-12
9:01 PM

0. Plan: Today: Preliminaries & asides, the cheese grater principle.
Next time: The actual proof.

1. Restate the theorem and the "detached wheels/hairy Y's" claim.

2. The Euler trick

a. **An Euler Interlude.** If you know brackets, how do you test exponentials? When's $e^Ae^B = e^Ce^D$?
Bad Idea. Take log and use BCH. You'll want to cry.
Clever Idea. Let E be the Euler derivation, which multiplies each element by its degree (e.g. on $\mathbb{Q}[[\phi]]$, $Ef = \phi\partial_\phi f$, so $Ee^\phi = \phi e^\phi$). Apply $\tilde{E}\zeta := \zeta^{-1}E\zeta$: $\tilde{E}(e^Ae^B) = e^{-B}e^{-A}(e^AAe^B + e^Ae^BB) = e^{-B}Ae^B + B = e^{-\operatorname{ad} B}(A) + B$.

From http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

b. The BCH formula

Thm. In $\widehat{FA}(x,y)$, $e^xe^y = e^\phi = e^{bch(x,y)}$, where ϕ is a Lie series.

* Good for making Lie groups out of Lie algebras.
* will be extremely important for us later on.

$\phi = \log e^xe^y$, but why is it a Lie series?

Proof. Apply $\tilde{E}$, get

$$\tilde{E}e^\phi = e^{-\operatorname{ad} y}(x) + y \in \widehat{Lie}(x,y)$$

Aside: how compute $\tilde{E}e^\phi$? In general, De^ϕ?

Aside^2: What's $de^A = d\exp$, where $\exp: M_{n\times n} \to$

$$e^{A+\epsilon B} = e^{A} + \epsilon \cdot ?_0 + O(\epsilon^2)$$

$$?_0 = \sum \frac{1}{n!} AA\dots B \dots\dots A \qquad BA = AB - \mathrm{ad}A(B)$$

$$= e^{A} \cdot F_{op}(e^{A}) = \qquad AAAA\cancel{A}^{B}$$

$$= e^{A} \cdot \frac{1-e^{-\mathrm{ad}A}}{\mathrm{ad}A}(B) \qquad \frac{e^{-\mathrm{ad}A}-1}{-\mathrm{ad}A}$$

So $\frac{1-e^{-\mathrm{ad}\,\phi}}{\mathrm{ad}\,\phi}(E\phi) = e^{-y}xe^{y}+y \in Lie(x,y)$

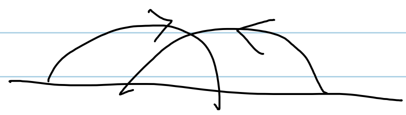
$$E\phi - \frac{1}{2}[\phi, E\phi] + \frac{1}{6}[\phi,[\phi,E\phi]] - \dots = e^{-\mathrm{ad}\,y}x + y$$

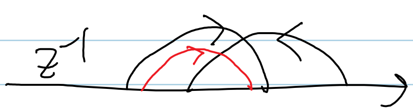
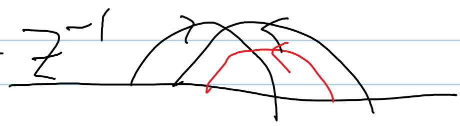
<u>Claim</u> ϕ exists and is unique and belongs to $Lie(x,y)$.

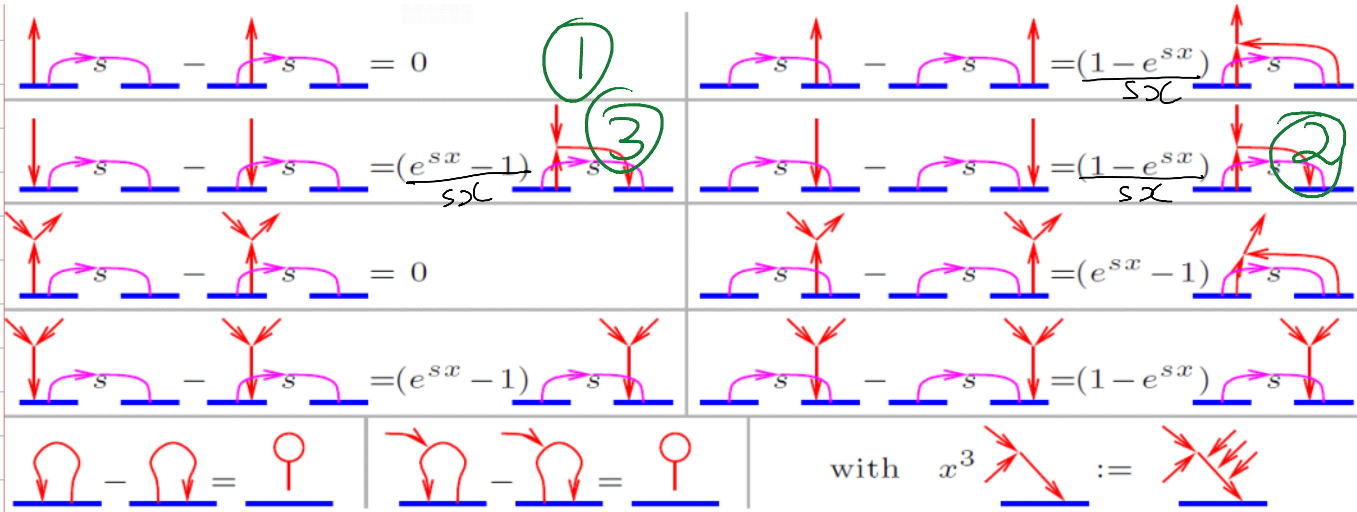
<u>Proof.</u> In the spirit of the inverse function thm, write $\phi_7 = \phi_6 + \varphi$......

done line

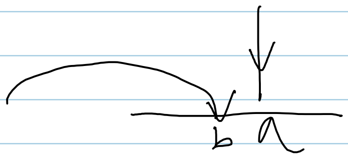
---

3. so apply $\tilde{E}$ to $z =$  & get

z^{-1}  $+ z^{-1}$ 



From http://www.math.toronto.edu/~drorbn/Talks/Toronto-1005/



$$e^{\mathrm{ad}\, b}(a) = e^b a e^{-b}$$

$$e^x = 1 + \frac{e^x - 1}{x} x$$

$$e^b a = e^{\mathrm{ad}\, b}(a) e^b = a e^b + \frac{e^{\mathrm{ad}\, b} - 1}{\mathrm{ad}\, b}([b,a]) \cdot e^b$$

---

4. The cheese-grater approach.