$O(3)$, $SO(3)$

$$\boxed{A^T A = Id}$$

you have done this computation!

1a) $A \in O(3) \Rightarrow A(S^2) = S^2 \Leftrightarrow \forall v \in \mathbb{R}^3 \quad \|Av\| = \|v\|$

(even a direct comput. works!)

1b) A direct computation is still doable!

$$A = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{33} \end{pmatrix}$$

$$x \to a_{11}x + \dots + a_{13}z$$
$$dy \to d(a_{21}x + \dots + a_{23}z)$$
$$dz \to d(a_{31}x + \dots + a_{33}z)$$

p.s. you know that

A is linear

η is a volume form on $\mathbb{R}^n$

$$A^*\eta = A^*(f\,dx_1 \dots dx_n) = (f \circ A)\det A \; dx_1 \dots dx_n$$

this does not directly apply to the volume form on S^2!

P.P.S

$$\omega = \iota^*(i_N\, dx \wedge dy \wedge dz)$$

$\iota : S^2 \to \mathbb{R}^3$

inclusion

1c) $\det B = 1$, $B \in SO(3)$

2a) $\underline{x^2+y^2+z^2=1}$

(Recall: $\begin{pmatrix} x: S^2 \to \mathbb{R} \\ y: S^2 \to \mathbb{R} \\ z: S^2 \to \mathbb{R} \end{pmatrix}$)

$$d(x^2+y^2+z^2) = d1 = 0$$

$$\| $$

$$2(x\,dx + \dots + z\,dz).$$

Hint: $x^2+y^2+z^2=1$

$$\underbrace{x^2+y^2 = 1-z^2}_{\text{doable using 2a)}}$$

2b) Try spherical coordinates!

$x = \cos\varphi \cos\theta$
$y = \cos\varphi \sin\theta$
$z = \sin\varphi$

Convert ω (from 1) to the spherical coord

You will get smth like

$$d(\sin\varphi)\wedge d\theta = -\underbrace{d\theta}\wedge \overset{\sin\varphi}{\overset{\|}{dz}}$$

$$\| \qquad\qquad\qquad \|$$

$$\cos\varphi\, d\varphi \wedge d\theta \qquad -\frac{x\,dy - y\,dx}{x^2+y^2}$$

2c) Hint: $\int_{S^2} \omega = ?$

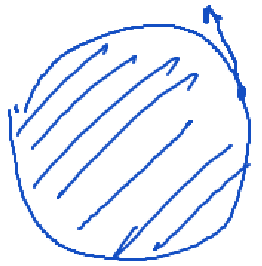
For any nice $U \subset S^2$

$$\int_U \omega = \text{Area}(U).$$

To get the cont, we integrate ω over  A

$$\int_A \frac{x\,dy - y\,dx}{x^2+y^2}\, dz = \int_A d\theta\, dz$$

Q3.



If we choose a "normal" field, we will end up with a consistently oriented

This is not quite well-defined, because there are a $\boxed{\text{lot}}$ of v. fields.

0) This is a local problem.

1) Idea: fix a local coord. system. $(x^1, x^2, \dots x^n)$ = coord. We can choose it in such a way, that the orient on the bndry looks like $\underbrace{\left(\frac{\partial}{\partial x^1}, \dots \frac{\partial}{\partial x^{n-1}}\right)}_{n-1 \text{ vectors!}}$

Add the vector field T

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & & & & T_1 \\ & 1 & 0 & & T_2 \\ 0 & & \ddots & & \vdots \\ & & & 1 & \\ \hline 0 & 0 & \dots & 0 & T_n \end{array}\right) \\ \frac{\partial}{\partial x^1} \dots \frac{\partial}{\partial x^n} \quad T \end{array}$$

v. field.

just a diff v. field

for $\underset{\sim}{T'}$ (another v. field)

$$\left(\begin{array}{ccc|c} 1 & & 0 & T_1' \\ & \ddots & & T_2' \\ 0 & & 1 & \vdots \\ \hline & 0 & & T_n' \end{array}\right)$$

T_n / T_n' a ~~stuff~~.

due to conv, $T_n < 0$, $T_n' < 0$.

$T_n / T_n' > 0$.

orient is same.

Q4. If det's are positive $\Rightarrow$ orient is consistent by def?

If M is orientable, for any point $p \in M$ choose a smooth chart. If neg. oriented, just replace x^k with $-x^k$.