Nov 3^{rd}, 11am-1pm }

Recall: $F(x,y): \mathbb{R}^{n+m} \to \mathbb{R}^m$, $F \in C^1$

$(x_1 \dots x_n, y_1 \dots y_m)$

$a = (x_0, y_0)$: $F(a) = 0$

$\det(DF|_y(y_0)) \neq 0$. Then $\exists f : \mathcal{U} \to \mathbb{R}^m$:

small nbhd of x_0

1) f is C^1

2) $F(x, f(x)) = 0$

Q1a)

$$a = (\underbrace{3}_{x}, \underbrace{-1, 2}_{y})$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

$DF|_y(-1,2)$

$$\det\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = 3 \neq 0$$

b) $g : \mathbb{R}^1 \to \mathbb{R}^2$

$$g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$$

$F(x, g_1(x), g_2(x)) = 0$

f is 0 for every x in some nbhd of 3

$$x \xrightarrow{h} \begin{pmatrix} x \\ g_1(x) \\ g_2(x) \end{pmatrix} \to F\begin{pmatrix} x \\ g_1(x) \\ g_2(x) \end{pmatrix}$$

$$0 = \frac{d}{dx} F(x, g_1, g_2) = \underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}}_{Df(3,-1,2)} \cdot \underbrace{\begin{pmatrix} 1 \\ g_1'(3) \\ g_2'(3) \end{pmatrix}}_{Dh(3)}$$

$$\begin{cases} 1 + 2g_1' + g_2' = 0 \\ 1 - g_1' + g_2' = 0 \end{cases}$$

a linear system

solve for g_1', g_2'

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & -1 & 1 \end{array}\right)$$

II

a) expressed (y, z) in terms of x

c) express (x, z) in terms of y I

(x, y) in terms of z II

$$\det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$$

Remark: can't apply IFT, this doesn't mean that the implicit fn-n doesn't exist.

$\boxed{\text{IFT is inconclusive}}$

(1-dim counterex: $x \to x^3$)

$$\det\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = -3 \neq 0$$

IFT holds $\Rightarrow$ implicit fn-n exists.

Q2 : $f: \mathbb{R}^2 \to \mathbb{R}$, $f \in C^2$ $\quad f(2,-1) = -1$

$$G(x,y,u) = f(x,y) + u^2$$
$$\underbrace{H(x,y,u) = ux + 3y^3 + u^3}$$

impl. fn-ns $\to g(-1) = 2$, $\to h(-1) = 1$

$$G(2,-1,1) = H(2,-1,1) = 0.$$

a) 1. express (x,u) in terms of y
$\quad\quad\quad\quad\quad\quad 1 \; 3$

2. Define $F(x,y,u) = \begin{pmatrix} G(x,y,u) \\ H(x,y,u) \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^2$

$$DF = \begin{array}{ccc} x & y & u \\ \end{array}$$
$$DF = \begin{pmatrix} \boxed{\frac{\partial f}{\partial x}} & \frac{\partial f}{\partial y} & \boxed{2u} \\ \boxed{u} & 9y^2 & \boxed{x + 3u^2} \end{pmatrix}$$

$$0 \neq \det \begin{pmatrix} \frac{\partial f}{\partial x} & 2u \\ u & x + 3u^2 \end{pmatrix}_{\substack{x = g(y) = 2 \\ y = -1 \\ z = h(y) = 1}} = \det \begin{pmatrix} \frac{\partial f}{\partial x}\Big|_{\substack{x=2 \\ y=-1}} & 2 \\ 1 & 5 \end{pmatrix}$$

$$5 \frac{\partial f}{\partial x}(2,-1) - 2 \neq 0 \iff \underbrace{\frac{\partial f}{\partial x}(2,-1) \neq 0.4}$$

Q4: $f: \mathbb{R}^{k+n} \to \mathbb{R}^n$ $f \in C^1$ Show: if c is close to 0, the $f(x) = c$ has a sol-n.

$f(0) = \underset{\sim}{0}$

$rk(Df(0)) = n$

not a square matrix!

$$Df(0) = \underbrace{\left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{n+k}} \\ \frac{\partial f_2}{\partial x_1} & & \frac{\partial f_2}{\partial x_{n+k}} \\ & \vdots & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_{n+k}} \end{pmatrix}\right\} n}_{n+k}$$

$f(x_1 \dots x_n, \overbrace{a_1 \dots a_k}^{\text{fixed}})$ use the inverse fn-n th-m.

Lin. alg. digression:

$A: \mathbb{R}^n \to \mathbb{R}^m$

1) $rkA = \dim(\mathrm{Im} A)$

2) $\underbrace{e_1, \dots e_n}_{\text{basis}} \Rightarrow A = \begin{pmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & \cdots & & \\ & & & a_{mn} \end{pmatrix}$

$rk_{row} A = \dim(\mathrm{span}(\text{column vecs}))$

3) $e_1 \dots e_n \Rightarrow A = \begin{pmatrix} a_{11} \dots & \\ & a_{nn} \end{pmatrix}$

$rk_{col} A = \dim(\mathrm{span}(\text{row vectors}))$

Lemma:

Choose n lin indep. column. $x_{i_1}, x_{i_2} \dots x_{i_n}$.

restrict $f|_{x_{i_1} \dots x_{i_n}}$. Then $\det(Df|_{x_{i_1} \dots x_{i_n}}) \neq 0$

$\exists g: U_0 \to \mathbb{R}^n$: $f(x, g(x)) = 0$ $\forall x \in U$.

$rkA = rk_{row}(A) = rk_{col}(A)$.

Q3: $\dfrac{\partial(f,g)}{\partial(x,y,z)} = DF = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \cdots \end{pmatrix}$ ← you only have to use the rank lemma

$F(x,y,z) = \begin{pmatrix} f(x,y,z) \\ g(x,y,z) \end{pmatrix}$

HW5 Q2a) $F: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} f(x,y) \\ y \end{pmatrix}$

the preim of $\begin{pmatrix} a \\ b \end{pmatrix}$ $\begin{pmatrix} a \\ b+\varepsilon/2 \end{pmatrix}$ are diff.

$F\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} f(x_1,y_1) \\ y_1 \end{pmatrix}$ (1)

$F\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} a \\ b+\varepsilon/2 \end{pmatrix} = \begin{pmatrix} f(x_2,y_2) \\ y_2 \end{pmatrix}$ (2)

$a = f(x_1, y_1)$ (1)

$a = f(x_2, y_2)$ (2)

Suppose: $\dfrac{\partial f}{\partial x}(x_0,y_0) \neq 0.$

$\det\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ 0 & 1 \end{pmatrix} = \dfrac{\partial f}{\partial x} \neq 0.$

IFT

F near (x_0,y_0) is a homeomorphism

$F(x_0,y_0) = \begin{pmatrix} a \\ b \end{pmatrix}$

$\exists (a-\varepsilon, a+\varepsilon) \times (b-\varepsilon, b+\varepsilon):$

$\forall (a',b') \in (a-\varepsilon, a+\varepsilon) \times \ldots$

$\exists (x', y'): F\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}$

the preim. of $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} a \\ b+\frac{\varepsilon}{2} \end{pmatrix}$ are differ.

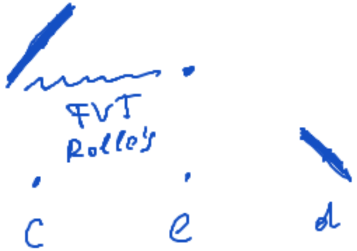
Thm: $f:[a,b]\to\mathbb{R}$

f is diffable on (a,b)

if $f'(c)>0$, $f'(d)<0$ for some $c,d\in(a,b) \Rightarrow \exists\, e: f'(e)=0$

$x^2\sin\left(\frac{1}{x}\right)$

Case 1: $f(c)>f(d)$



$f(e)=f(c)$

Rolle's.

Use the thm to prove that no $c,d\in\mathbb{R}$:

$f'(c)>0$

$f'(d)<0$.

Can't have this!

$\forall a\in\mathbb{R}$

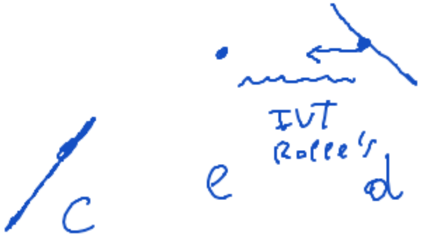
$f'(a)>0 \Rightarrow f$ is strictly inc.

or

$\forall a\in\mathbb{R}$

$f'(a)<0 \Rightarrow f$ is str. decr.

Case 2: $f(c)<f(d)$



$\exists e:$ $f(e)=f(d)$

Rolle's

What we have proved:

$$\left.\begin{array}{l} f'(a)>0 \\ f'(b)<0 \end{array}\right\} \Rightarrow f'(c)=0 \qquad \boxed{\text{Rolle's}}$$

Corollary: $f'(a) < f'(b) \Rightarrow \forall y \in [f'(a), f'(b)]$

$\exists c \in [a,b] : f'(c) = y$

Idea: instead of f consider $f(x) - x\cdot y = g'(x)$.

$g'(a) = f'(a) - y < 0$

$g'(b) = f'(b) - y > 0$

$\exists c : g'(c) = \underbrace{0 = f'(c) - y}_{f'(c)=y}$

$$\lim_{(h,k)\to 0} \frac{|f(h,k)|}{\sqrt{\|h\|^2+\|k\|^2}} = 0$$

$\forall x \in \mathbb{R}$

$$\lim_{(h,k)\to 0} \frac{|f(h,k)|}{\sqrt{\|h\|^2+\|k\|^2}} = \lim_{(h,k)\to 0} \frac{|f(xh,xk)|}{|x|\cdot\sqrt{\|h\|^2+\|k\|^2}} =$$

$$= |x| \cdot \lim \frac{|f(h,k)|}{\sqrt{\|h\|^2+\|k\|^2}}$$

$(h,k) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\left\| \frac{(h,k)}{\|(h,k)\|} \right\| = 1$$

$$\sqrt{\|h\|^2 + \|k\|^2}$$

$$\boxed{\|h\| + \|k\|}$$

$$\lim_{(h,k)\to 0} \left| \frac{f(h,k)}{\|(h,k)\|} \right| = \left| f\left(\frac{h}{\sqrt{\|(h,k)\|}}, \frac{k}{\sqrt{\|(h,k)\|}} \right) \right|$$

$$\lim \left| f\left(\frac{h}{\sqrt{\ldots}}, \frac{k}{\sqrt{\ldots}} \right) \right| =$$

$$\left| f\left(\lim_{(h,k)\to 0} \left(\frac{h}{\sqrt{\|(h,k)\|}}, \frac{k}{\sqrt{\|(h,k)\|}} \right) \right) \right|$$

$$\left\| \frac{h}{\sqrt{\|(h,k)\|}} \right\| = \frac{\|h\|}{\sqrt{\|(h,k)\|}} =$$

$$= \frac{1}{\sqrt{\frac{\|(h,k)\|}{\|h\|^2}}} \to \infty$$

$\|h\| \sim 10^{-5}$ $\quad \|(h,k)\| \sim 10^{-5}$

$\|k\| \sim 10^{-5}$ $\quad \sqrt{\|(h,k)\|} \sim 10^{-2,5}$