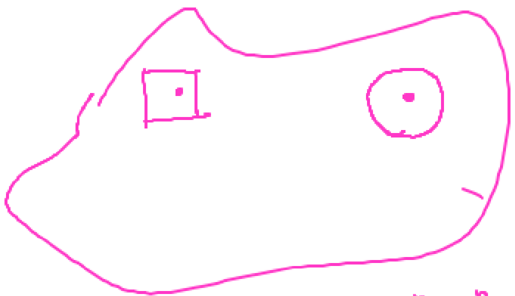
Remark: compact sets   covers of sets

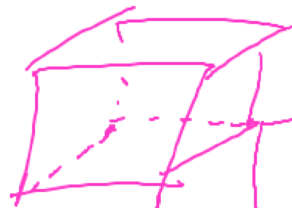
open sets

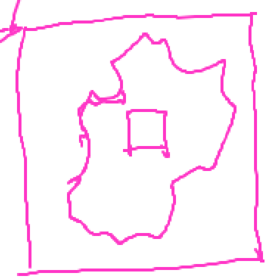
A set $\mathcal{U}$ in $\mathbb{R}^n$ is open $\iff$ $\forall x \in \mathcal{U}$ is cont. with a open rectangle $x \in R \subseteq \mathcal{U}$.

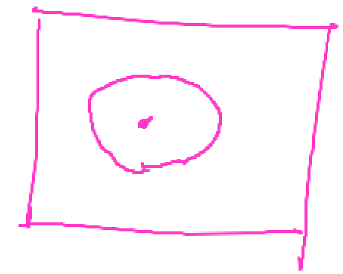
a rectangle:

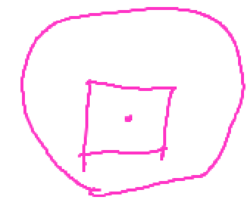


diff def-ns









Exer: find c_1, c_2, D_1, D_2 s.t.

$$\begin{cases} C_1\|\cdot\|_2 \le \|\cdot\|_1 \le C_2\|\cdot\|_2 \\ D_1\|\cdot\|_1 \le \|\cdot\|_2 \le D_2\|\cdot\|_1 \end{cases}$$

$|x_1 - y_1| < \varepsilon_1$
;
$|x_n - y_n| < \varepsilon_n$

$\|x - y\|_2 < \varepsilon$.

Prop: all def-ns stay the same if rect $\longleftrightarrow$ balls

Exer: prove rig. in $\mathbb{R}^n$

$\|\cdot\|_1 \sim \|\cdot\|_2$

The topologies def. by rect. and by balls are equiv.

Q1

A_1

open or closed? ball ← intuitive

A_2

sphere in $\mathbb{R}^n$ ← intuitive

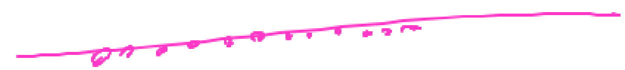
"n-sphere" (I think that n-sphere lies in $\mathbb{R}^{n+1}$)

← "n-1-sphere"

A_3

"dense" set

$\mathbb{R}^1$



$\mathbb{Q}$ is dense $\forall (\;\;)$ has an irrat + rational

$\mathbb{R} \setminus \mathbb{Q}$ is dense

Q2

A

A is closed

x can be separated from A.

cont. in metric sense

Idea: define $f_x : \mathbb{R}^n \to \mathbb{R}$, $f_x(y) = |x-y|$

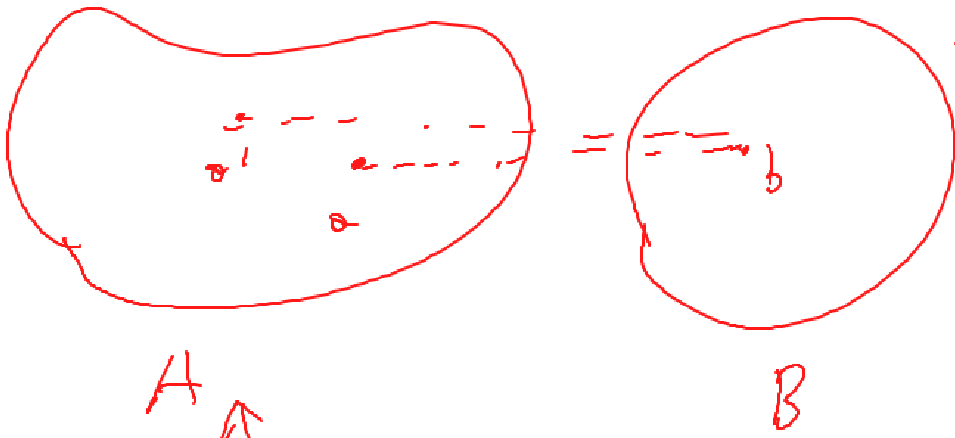
nice

more techn. but more useful

f_x is a cont. f-n., the preimage of an open set is open.

preim. of closed sets

Q2b)



doesn't need to be comp.

does

no need for a fact like this → $A \cap B = \emptyset$

Good Q about Hausd. distance: is there an $a' \in A$ s.t. $|a'-b| = \inf_{a\in A} |a-b|$

$\exists d>0: \ |x-y| \geq d \ \forall x \in A, \ y \in B$

think about it

Define $\forall b \in B$

0) inf $< \infty$ – $\inf_{a \in A} |a-b| := f(b)$

1) f is a cont. on B

2) $f > 0$. $\Leftarrow$ Q2a

this breaks in Q2c)

$\forall$ ⊙ ⊙

Q3 holds for every Hausd. top. space.

Exer: triangle ineq

$\Downarrow$

every metric space is Hausd.

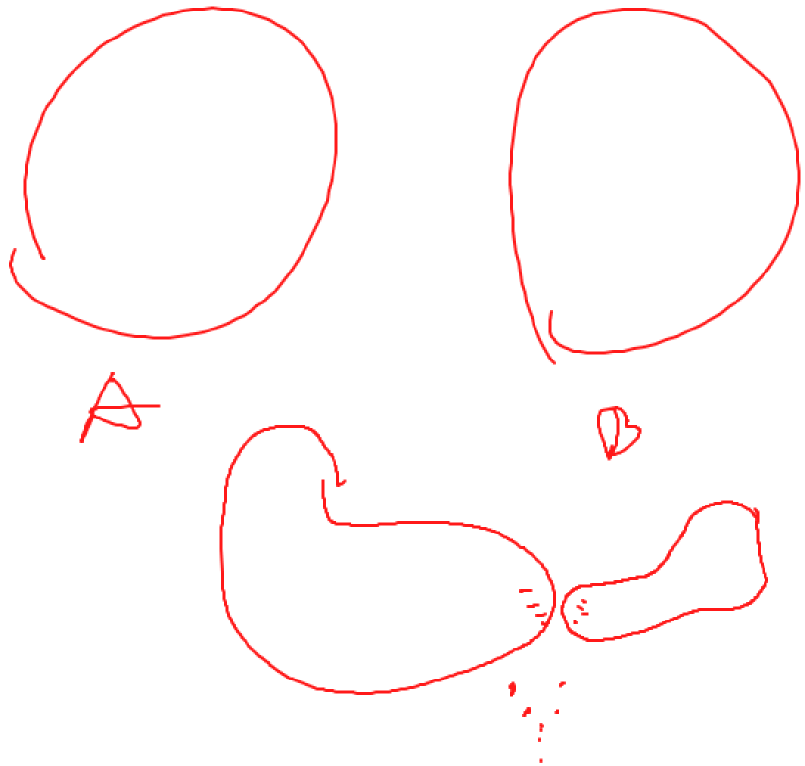
Q2b $\Leftrightarrow$ inf > 0 ~~every metric~~

Q2c says that

in Q2c) $A \cap B = \emptyset$ are closed

(both non-comp)

required to be non-comp.

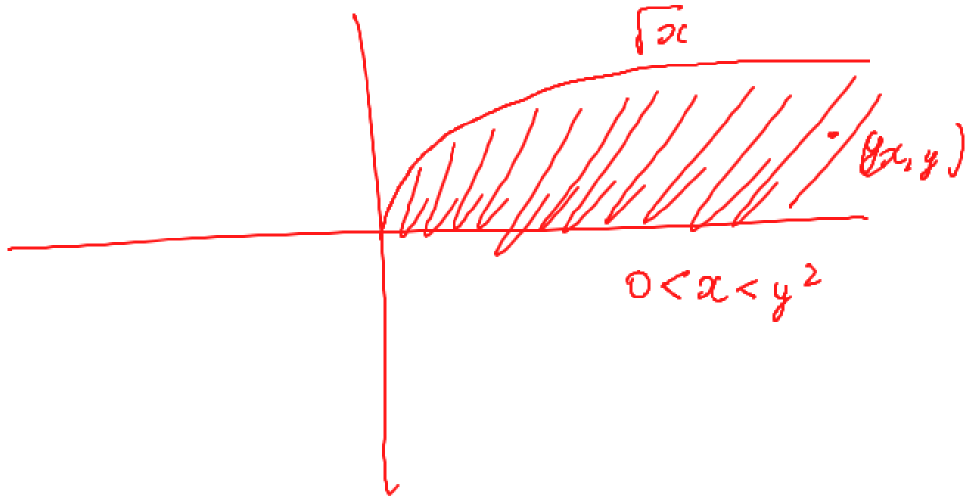


$\Rightarrow \inf \overset{\text{might}}{=} 0$

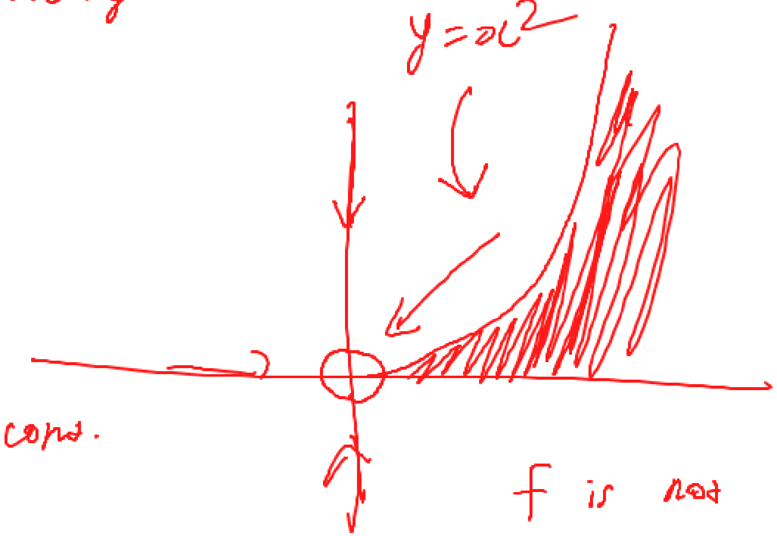
$\cancel{A \cap B \neq \emptyset \Rightarrow \inf = 0}$

Q5:

A



Correct



$0<y<x^2$

Q5

diff limits

To get a contrad. go under param. of this point!

discont

this argument proves that f is discont.

prove this

$\boxed{\text{every straight line "exits" } A \text{ at some point}}$

f is not cont on $(0,0)$

one-point set

Notice that $f^{-1}(\{1\})$ is closed.

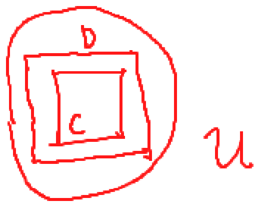
$f^{-1}(\{1\}) = A$

contrad.

if f is cont $\Rightarrow$ A is closed due to being a preim

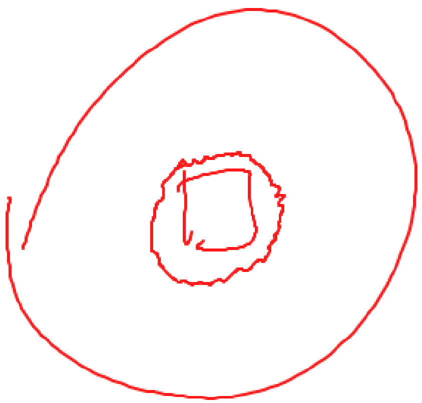
Q3

$C \subseteq \text{Int } D$



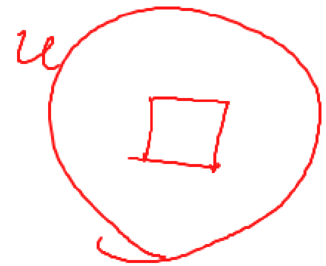
Weird idea: try to see why $C = D$ doesn't work?

C and D are compact in the induced top of $\mathcal{U}$



compact sets in Hausd. ($\mathbb{R}^n$) are closed

take an open nbhd and closure



Closure has to be in $\mathcal{U}$

Q4: Last assignment

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$\|Tx\|_m \leq C \, \|x\|_n$$

bounded (takes bounded sets to bounded)

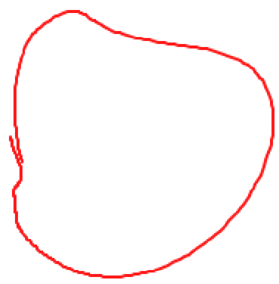
reliable

$$\Downarrow$$

T is continuous. (ε–δ 100% works!)

(resp. using preimages)

Q6:



K

K is comp $\Longrightarrow$ (any cont. f-n on K is bounded)

reverse impl. holds!

$\Longleftarrow$

not in K

f-n which is unbounded

cont. everywh. except 1 point ($\mathbb{R}^n$)

counter ex for $\mathbb{R}^2$ is well-known!