MAT257 Term Test 1 Rejects

The following questions were a part of a question pool for the 2020-21 MAT257 Term Test 1, but at the end, they were not included.

1. If A is an interval, its closure is equal to the closure of its interior. Is this true for all subsets of $\mathbb{R}$?

2. Prove that if $C \subset \mathbb{R}^n$ is a compact set and $f\colon C \to \mathbb{R}^m$ is continuous, then the graph of f, the set $\Gamma(f) := \{(x, f(x))\colon x \in C\}$, is compact.

3. Let $A = (a_{ij})$ be an $n \times n$ matrix, let $f\colon \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \sum a_{ij}x_ix_j$, and let $y \in \mathbb{R}^n$. Find $Df(y)$.

4. We will say that a function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is "uniformly ϵ-near λ", where λ is an invertible linear transformation, if $f(0) = 0$ and
$$\forall x_1, x_2 \in \mathbb{R}^n, \qquad |\lambda(x_1 - x_2) - (f(x_1) - f(x_2))| \leq \epsilon|x_1 - x_2|.$$
Prove that if $\epsilon > 0$ is sufficiently small, then for every $y \in \mathbb{R}^n$ there is some $x \in \mathbb{R}^n$ such that $f(x) = y$.

5. A function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is tiny; namely, $f \in o(h)$. Show that f is differentiable at 0. What is $f'(0)$?

6. Define $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (e^x + e^y, e^x + e^{-y})$.
   (a) Explain why every point $(a, b) \in \mathbb{R}^2$ has a neighborhood on which f is invertible.
   (b) If $f(a, b) = (c, d)$, compute the differential of f^{-1} at (c, d).

7. We consider the equation $x + y + z = \sin xyz$ very near $x = y = z = 0$, and we want to solve for z as a function of x and y. Show that this can be done and compute the partial derivatives of the solution with respect to x and to y.

8. Determine the interiors, exteriors, and boundaries of the subsets of $\mathbb{R}^2$ given below:
   (a) $A = \{(x, y)\colon x, y \in \mathbb{Q}\}$.
   (b) $B = \{(x, y)\colon 0 < x^2 + y^2 < 1\}$.
   (c) $C = \{(x, y)\colon y < x^2\}$.
   (d) $D = \{(x, y)\colon y \leq x^2\}$.

9. Show that there is no neighborhood of 0 in which the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = |xy|$ has continuous partial derivatives. Yet show that this f is differentiable at 0.

10. A function $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ satisfies $f(0) = (1, 2)$ and $f'(0) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, and a function $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by $g(x, y) = (x + 2y + 1, 3xy)$. Compute $(g \circ f)'(0)$.

11. For some values of n and m, give an example of a continuously differentiable function $\gamma\colon \mathbb{R}^n \to \mathbb{R}^m$ whose differential is 1-1 for every $a \in \mathbb{R}^n$, yet such that γ itself is not 1-1.

12. Determine the interiors, exteriors, and boundaries of the subsets of $\mathbb{R}^2$ given below:

    (a) $A = \{(x, y)\colon\ x = 0\}$.

    (b) $B = \{(x, y)\colon\ 0 \le x < 1\}$.

    (c) $C = \{(x, y)\colon\ 0 \le x < 1 \text{ and } 0 \le y < 1\}$.

    (d) $D = \{(x, y)\colon\ x \in \mathbb{Q} \text{ and } y > 0\}$.

13. A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is called "rotation invariant" if for every θ, $f \circ T_\theta = f$, where $T_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation given (in the standard basis) by the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. If a given $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable and rotation invariant, show that $Df(x, y)(-y, x) = 0$ for every $(x, y) \in \mathbb{R}^2$.