Q1

$\Lambda^1(\mathbb{R}^3)$ is just the dual space of $\mathbb{R}^3$, which we will assign a dual basis $(\varphi_1, \varphi_2, \varphi_3)$. It has dimension 3, so is isomorphic to $\mathbb{R}^3$. An isomorphism to $\mathbb{R}^3$ is given by

$$a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3 \mapsto (a_1, a_2, a_3).$$

This is an isomorphism because every triple $(a_1, a_2, a_3) \in \mathbb{R}^3$ corresponds to some $a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3 \in \Lambda^1(V)$, and vice versa. Also, $\Lambda^2(\mathbb{R}^3)$ has dimension $\binom{3}{2} = 3$, with basis $(\varphi_2 \wedge \varphi_3, \varphi_3 \wedge \varphi_1, \varphi_1 \wedge \varphi_2)$. (Conventionally we would use $\varphi_1 \wedge \varphi_3$, but this is equal to $-\varphi_3 \wedge \varphi_1$ so it is equivalent.) We will use the isomorphism

$$a_1(\varphi_2 \wedge \varphi_3) + a_2(\varphi_3 \wedge \varphi_1) + a_3(\varphi_1 \wedge \varphi_2) \mapsto (a_1, a_2, a_3)$$

to $\mathbb{R}^3$. Then

$$\begin{aligned}(a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3) \wedge (b_1\varphi_1 + b_2\varphi_2 + b_3\varphi_3) &= (a_2b_3 - a_3b_2)\varphi_2 \wedge \varphi_3 + (-a_1b_3 + a_3b_1)\varphi_3 \wedge \varphi_1 \\ &\quad + (a_1b_2 - a_2b_1)\varphi_1 \wedge \varphi_2,\end{aligned}$$

which is mapped to $(a_2b_3 - a_3b_2, -a_1b_3 + a_3b_1, a_1b_2 - a_2b_1)$ in $\mathbb{R}^3$ with the above isomorphism. This is precisely the cross product $(a_1, a_2, a_3) \times (b_1, b_2, b_3)$ in $\mathbb{R}^3$.

Q2

It was shown in class that L is orientation-preserving if and only if $\det(L) > 0$.

a) The matrix of L_1 is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, which has $\det(L_1) = -1$, so it is orientation-reversing.

b) $L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which has $\det(L_2) = -1$, so L_2 is orientation-reversing.

c) For (c) and (d), note the general rotation matrix by θ is $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, with determinant $\cos^2\theta + \sin^2\theta = 1$, so all rotation matrices are orientation-preserving. Thus L_3 is orientation-preserving.

d) By the above, L_4 is orientation-preserving.

e) By identifying $(a, b) \in \mathbb{R}^2$ with $a + bi \in \mathbb{C}$, the matrix of L_5 is $L_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. $\det(L_5) = -1$, so it is orientation-reversing.

f) $L_6 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, which has $\det(L_6) = 1$, so L_6 is orientation-preserving.

g) L_7 is just -1 times the identity matrix. Thus, its determinant is $(-1)^n$, so L_7 is orientation-preserving if n is even and orientation-reversing if n is odd.

h) The matrix of L_8 is

$$L_8 = \left[\begin{array}{c|c} 0 & I_n \\ \hline I_m & 0 \end{array}\right],$$

where I_n and I_m are the $n \times n$ and $m \times m$ identity matrices, respectively, and 0 is a matrix of zeros. Note that permuting the rows of L_8 by a suitable permutation $\sigma \in S_{m+n}$ results in the identity matrix, which has determinant 1. Since the determinant is an alternating tensor, permuting the rows by σ will change the sign of $\det(L_8)$ by $\text{sign}(\sigma)$, so $\det(L_8) = \text{sign}(\sigma)$. The desired σ is a $m+n$ cycle applied m times "upward" to shift I_m to the top. An $m+n$ cycle has sign $(-1)^{m+n+1}$, so

$$\begin{aligned} \text{sign}(\sigma) &= ((-1)^{m+n+1})^m \\ &= (-1)^{m^2+m+mn} \\ &= (-1)^{mn}. \end{aligned}$$

The last line follows because either m or $m+1$ is even, so $m^2 + m = m(m+1)$ is always even. Thus, $\text{sign}(\sigma) = (-1)^{mn}$, so L_8 is orientation-preserving if mn is even and orientation-reversing if mn is odd.

Q3

Let $\phi \in \Lambda^k(V)$. Define $\psi_k : \Lambda^{n-k}(V) \to \left(\Lambda^k(V)\right)^*$ by $\psi_k(\lambda) = \chi(\phi \wedge \lambda)$. This is a linear map because χ is a linear map and the wedge product is bilinear, so their composition will be linear. In addition,

$$\begin{aligned} \dim((\Lambda^k(V))^*) &= \dim(\Lambda^k(V)) \\ &= \binom{n}{k} \\ &= \binom{n}{n-k} \\ &= \dim(\Lambda^{n-k}(V)), \end{aligned}$$

so showing ψ_k is injective is enough to prove it is an isomorphism. Suppose $\psi_k(\lambda_1) = \psi_k(\lambda_2)$, so $\chi(\phi \wedge \lambda_1) = \chi(\phi \wedge \lambda_2)$ for all $\phi \in \Lambda^k(V)$. Since χ is an isomorphism, this means $\phi \wedge \lambda_1 = \phi \wedge \lambda_2$, or $\phi \wedge (\lambda_1 - \lambda_2) = 0$ for all ϕ. Writing $\lambda_1 - \lambda_2 = \sum_I a_I \omega_I$ using in the standard basis ω_I, we see that if any $a_I \neq 0$, we could take $\phi = \omega_I$ and then $\phi \wedge (\lambda_1 - \lambda_2) = a_I \neq 0$. Thus, $\lambda_1 = \lambda_2$, so ψ_k is injective.

Q4

a) For $I \in \underline{n}_a^k$, define I^c as the "complement" of I, i.e. the unique multi-index in $\underline{n}_a^{n-k}$ such that I and I^c share no indices. It is unique because the lengths of I and I^c sum to n, and they are both required to be increasing. Also, define $I + J$ to be the multi-index obtained by concatenating I and J. Let ω_I be the elementary alternating tensors with multi-index I. Then, we claim that $*(\omega_I) = (-1)^\sigma \omega_{I^c}$, where $\sigma \in S_n$ is the permutation that reorders $I + I^c \in \underline{n}^n$ to be increasing, is the desired isomorphism. This is an isomorphism because every $J \in \underline{n}_a^{n-k}$ is uniquely determined by $J^c \in \underline{n}_a^k$ and vice versa.
We now evaluate $\omega_I \wedge (*\omega_J)$. If $J = I$, then $\sigma(I + I^c) = (1, 2, \dots, n)$ for some σ, and

$$\begin{aligned}\omega_I \wedge (*\omega_I) &= (-1)^\sigma \omega_I \wedge \omega_{I^c} \\ &= ((-1)^\sigma)^2 \omega_n \\ &= \omega_n,\end{aligned}$$

since reordering the φ_i to be increasing introduces a factor of $(-1)^\sigma$. If $J \neq I$, ω_{J^c} will share some φ_k with ω_I, and their wedge product will be 0. Thus,

$$\omega_I \wedge (*\omega_J) = \delta_{IJ}\omega_n = \langle \omega_I, \omega_J \rangle.$$

This is enough to show that $\lambda \wedge (*\eta) = \langle \lambda, \eta \rangle \omega_n$ for arbitrary alternating tensors, since expanding them in the basis gives

$$\begin{aligned}\left(\sum_I a_I \omega_I\right) \wedge \left(\sum_J b_J * (\omega_J)\right) &= \sum_{I,J} a_I b_J \omega_I \wedge (*\omega_J) \\ &= \sum_I a_I b_I \omega_n \\ &= \langle \lambda, \eta \rangle \omega_n.\end{aligned}$$

Finally, this isomorphism is unique, because if $\lambda \wedge (*\eta) = \langle \lambda, \eta \rangle \omega_n$ for all $\lambda, \eta \in \Lambda^k(V)$, we must have

$$\sum_{I,J} a_I b_J \omega_I \wedge (*\omega_J) = \sum_I a_I b_I \omega_n,$$

so comparing terms gives $\omega_I \wedge (*\omega_I) = \omega_n$ for all I. The only way to do this is with the definition of $*$ given above, because mapping $*(\omega_I)$ to any other ω_J would result in a repetition in the wedge product, causing it to be 0.

b) On the left is the original tensor, and on the right is what it gets mapped to by $*$.

ω_1	ω_{23}
ω_2	$-\omega_{13}$
ω_3	ω_{12}

ω_{12}	ω_{34}
ω_{13}	$-\omega_{24}$
ω_{14}	ω_{23}
ω_{23}	ω_{14}
ω_{24}	$-\omega_{13}$
ω_{34}	ω_{12}

$(2,1,3)$ and $(1,3,2,4)$ both require one transposition to order, and $(2,4,1,3)$ requires three transpositions, so σ is odd and the sign is negated. The rest require an even number of transpositions, so the sign is unaffected.

c) Since $*$ is linear, it suffices to show $* \circ *(\omega_I) = (-1)^{k(n-k)}$ for the elementary alternating tensors ω_I. We will denote $* \circ *$ by $*^2$. We see that

$$\begin{aligned} *^2(\omega_I) &= *((-1)^{\sigma_1}\omega_{I^c}) \\ &= (-1)^{\sigma_1}(-1)^{\sigma_2}\omega_I \\ &= (-1)^N\omega_I \end{aligned}$$

since $(I^c)^c = I$, so $*^2$ is a scalar multiple of the identity map. Thus,

$$\begin{aligned} (*^2\omega_I) \wedge (*\omega_I) &= (-1)^N\omega_n \\ &= (-1)^{k(n-k)}(*\omega_I) \wedge (*^2\omega_I) \\ &= (-1)^{k(n-k)}\omega_n, \end{aligned}$$

where the second line follows from the supercommutative property of the wedge product, and the last line follows from the definition of $*$ applied to $*(\omega_I)$. Comparing lines 1 and 3, we see that $N = k(n-k)$, so $*^2(\omega_I) = (-1)^{k(n-k)}\omega_I$.