Q1

Note that if $u \in \mathbb{R}^4$ and $\omega \in \Lambda^2(\mathbb{R}^4)$, then $\omega(u, u) = 0$.

a) Let $x = (1, 0, 0, 0)$. We have $f(x, x) = x_1x_2 - x_2x_1 + x_1^2 = 1 \neq 0$, so f is not alternating.

b) Let $x = (1, 0, 1, 0)$. We have $g(x, x) = x_1x_3 - x_3x_2 = 1 \neq 0$, so g is not alternating.

c) If h is a tensor, it should satisfy $h(2x, y) = h(x, y) + h(x, y) = 2h(x, y)$. Consider $x = (1, 0, 0, 0)$ and $y = (0, 1, 0, 0)$. Then $h(x, y) = 1$, and $h(x, y) + h(x, y) = 2$, but $h(2x, y) = 8$, which is not equal to $2h(x, y)$. Thus, h cannot be an alternating tensor since it is not a tensor.

Q2

The determinant of a matrix in $\mathbb{R}^{n\times n}$ can be regarded as an alternating tensor on $(\mathbb{R}^n)^n$, i.e. $\det \in \Lambda^n(\mathbb{R}^n)$. We know that $\dim(\Lambda^n(\mathbb{R}^n)) = \binom{n}{n} = 1$, since the only increasing permutation in $\underline{n}^n$ is the identity, which we will denote by id. Thus, we may write the determinant in the basis of length 1 of $\Lambda^n(\mathbb{R}^n)$, i.e.

$$\det(v_1, \dots, v_n) = a \sum_{\sigma \in S_k} (-1)^\sigma (\varphi_{\text{id}} \circ \sigma)(v_1, \dots, v_n),$$

where $a \in \mathbb{R}$ is to be determined. Let $(v_1, \dots, v_n)$ be the standard basis of R^n. To find a, it suffices to know the value of det for one particular list of vectors $x_I = (v_{i_1}, \dots, v_{i_n})$. Again, the only possible choice of I is the identity permutation $(1, 2, \dots, n)$, and this corresponds to the identity matrix, which has determinant 1. Hence,

$$\begin{aligned} 1 &= a \sum_{\sigma \in S_k} (-1)^\sigma (\varphi_{\text{id}} \circ \sigma)(v_1, \dots, v_n) \\ &= a(-1)^{\text{id}} \varphi_{\text{id}}(v_1, \dots, v_n), \end{aligned}$$

since by definition $\varphi_I(v_J) = \delta_{IJ}$. Note that $\varphi_{\text{id}}(v_1, \dots, v_n)$ is the product of all the i^{th} components of v_i, which is 1 since $(v_1, \dots, v_n)$ is the standard basis. Moreover, the sign of id is 1, so we get $a = 1$. It follows that

$$\det = \sum_{\sigma \in S_k} (-1)^\sigma (\varphi_{\text{id}} \circ \sigma) = \omega_{\text{id}}.$$

Q3

Let $(v_1, \dots, v_m)$ be the standard basis of $\mathbb{R}^m$ and $x_L = (v_{l_1}, \dots, v_{l_k})$ for $L \in \underline{n}_a^k$. We have that $L^*\omega_I = \omega \circ A$, so

$$(\omega_I \circ A)x_L = \sum_{J \in \underline{m}_a^k} c_J \omega_J(x_L).$$

By properties of elementary alternating $k-$tensors, $\omega_J(x_L) = \delta_{JL}$, so the right hand side is c_L. Now let $\alpha_i \in \mathbb{R}^n$ be the i^{th} column of A, and note that $Av_i = \alpha_i$ since v_i is part of the standard basis of $\mathbb{R}^m$. The left hand side is therefore

$$\begin{aligned}
\omega_I(Av_{l_1}, \dots, Av_{l_k}) &= \omega_I(\alpha_{l_1}, \dots, \alpha_{l_k}) \\
&= \sum_{\sigma \in S_k} (-1)^\sigma (\varphi_I \circ \sigma)(\alpha_{l_1}, \dots, \alpha_{l_k}) \\
&= \sum_{\sigma \in S_k} (-1)^\sigma \varphi_I(\alpha_{\sigma(l_1)}, \dots, \alpha_{\sigma(l_k)}) \\
&= \sum_{\sigma \in S_k} (-1)^\sigma A_{i_1,\sigma(l_1)} \cdots A_{i_k,\sigma(l_k)},
\end{aligned}$$

since φ_I picks the i_1^{th} element of $\alpha_{\sigma(l_1)}$, namely, the entry $(i_1, \sigma(l_1))$ in A, and so on. Thus,

$$c_J = \sum_{\sigma \in S_k} (-1)^\sigma A_{i_1,\sigma(j_1)} \cdots A_{i_k,\sigma(j_k)},$$

where $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$.
Remark: we can also say that c_J is the $k \times k$ minor of A corresponding to the rows $(i_1, \dots, i_k)$ and the columns $(j_1, \dots, j_k)$.

Q4

Let $\underline{n}_s^k = \{(i_1, i_2, \dots, i_k) \in \underline{n}^k : i_1 \leq i_2 \leq \dots \leq i_k\}$, i.e. the subset of $\underline{n}^k$ consisting of only non-decreasing sequences. We claim that the collection of σ_I where $\sigma_I = \sum_{\tau \in S_k} \varphi_I \circ \tau$ and $I \in \underline{n}_s^k$ is a basis for $S^k(V)$, with dimension $\binom{n+k-1}{k}$.

First, we show that σ_I is symmetric, which is equivalent to showing $\sigma_I \circ \eta = \sigma_I$ for $\eta \in S_k$. We have

$$\begin{aligned}\sigma_I \circ \eta &= \sum_{\tau \in S_k} (\varphi_I \circ \tau) \circ \eta \\ &= \sum_{\tau \in S_k} \varphi_I \circ (\tau \circ \eta) \\ &= \sum_{\tau' \in S_k} \varphi_I \circ \tau' = \sigma_I,\end{aligned}$$

where the last line is justified because right multiplication by an element in S_k is a bijection from S_k to S_k. Next, we show that we can express any $T \in S^k(V)$ as a sum of σ_I, i.e. $T = \sum_{I \in \underline{n}_s^k} b_I \sigma_I$, where $b_I \in \mathbb{R}$. Let $(v_1, \dots, v_n)$ be a basis for V. Note that we can write any $I \in \underline{n}_s^k$ uniquely as a n-tuple $a_I = (a_1, \dots, a_n)$, where a_i is the number of times that i appears in I, because I is nondecreasing. Conversely, an n-tuple $(a_1, \dots, a_n)$ with $a_1 + \dots + a_n = k$ uniquely specifies some nondecreasing sequence I. Then, evaluating T on v_J gives

$$\begin{aligned}T(v_J) &= \sum_{I \in \underline{n}_s^k} b_I \sigma_I(v_J) \\ &= b_J a_1! \cdots a_n!,\end{aligned}$$

because $(\sigma_I \circ \tau) v_J = 0$ unless there exist permutations $\tau \in S_k$ that leave the sequence J unchanged. This occurs if and only if τ permutes the repeated entries, which can happen in $a_1! \cdots a_n!$ ways (the 1s can be permuted $a_1!$ ways, etc.). Hence,

$$T = \sum_{I \in \underline{n}_s^k} b_I \sigma_I,$$

where $b_I = \frac{1}{a_1! \cdots a_n!}$ and $(a_1, \dots, a_n)$ is the n-tuple corresponding to I. Finally, the dimension of $S^k(V)$ is given by the number of nondecreasing sequences in $\underline{n}^k$. But this was shown in class to be $\binom{n+k-1}{k}$.