Q1

Define $A = (0,a) \times (0, \frac{\pi}{2}) \times (0, 2\pi)$. It's pretty clear that $G(A) = V - \{(x,y,z) \in V : x = y = 0\}$, since the square of the norm of anything in $g(A)$ is just r^2, and since $r < a$, $r^2 < a^2$, and $z = r\sin\phi > 0$, however x and y are 0 only when $\cos\phi = 0$, which never happens. Meaning that anything on the z axis is not in $g(A)$.

Conversely, if (x,y,z) is in $V - \{(x,y,z) \in V : x = y = 0\}$, then taking $r = \sqrt{x^2+y^2+z^2}$, $\phi = \arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)$, $\theta = \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)$, which is indeed an ordered triple in A gives us (x,y,z) back. So, it is indeed true that everything in $V - \{(x,y,z) \in V : x = y = 0\}$ is in $g(A)$. Finally, since all points in $g(A)$ are in the form given above, it is indeed true that $g(A) = V - \{(x,y,z) \in V : x = y = 0\}$.

Now, the section of the z axis in V (the set missing, $\{(x,y,z) \in V : x = y = 0\}$) is bounded and of measure 0 in $\mathbb{R}^3$, so it of content 0. Thus, for any integrable f, $\int_{g(A)} f = \int_V f$.

Finally, note that g is clearly injective on A. Each component function is not, but if you change $r, \theta, or \phi$, at least one of the component functions will be different (since A only contains θ, ϕ on less than 1 period), and thus g i injective. Thus, $g : A \to g(A)$ is 1-1.

Now, we want to use the change of variables theorem on:

$$\int_V f = \int_{g(A)} f$$

and then change of variables tells us:

$$\int_{g(A)} f = \int_A f \circ g |\det g'|$$

Using change of variables, we get:

$$\int_V z = \int_A z \circ g |\det g'| = \int_A r\sin\phi |\det g'|$$

So, we just need to know the Jacobian determinant of g.

$$g' = \begin{pmatrix} D_1g_1 & D_2g_1 & D_3g_1 \\ D_1g_2 & D_2g_2 & D_3g_2 \\ D_1g_3 & D_2g_3 & D_3g_3 \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\theta & -r\sin\phi\cos\theta & -r\cos\phi\sin\theta \\ \cos\phi\sin\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi & r\cos\phi & 0 \end{pmatrix}$$

Which has determinant (using the bottom row so that I can take advantage of that 0)

$$-\sin\phi\left[r^2\sin\phi\cos\phi\cos^2\theta + r^2\sin\phi\cos\phi\sin^2\theta\right] - r\cos\phi\left[r\cos^2\phi\cos^2\theta + r\cos^2\phi\sin^2\theta\right] + 0 = -r^2\cos\phi$$

(Turns out a lot cancels out). So, then we have:

$$\int_A r\sin\phi |\det g'| = \int_A r^2\sin\phi\cos\phi$$

This function is obviously continuous, so Fubini tells us:

$$\int_A r^2\sin\phi\cos\phi = \int_0^a \int_0^{\frac{\pi}{2}} \int_0^{2\pi} r^2\sin\phi\cos\phi \, d\theta \, d\phi \, dr$$

and we're done.

Q2

Suppose for the sake of contradiction that $\int_{U_1} f$ exists. Now we're gonna do some change of variables stuff. Define $g(r,\theta) = (r\cos\theta, r\sin\theta)$. Then, consider the set $A = (0,1) \times (0, 2\pi)$. It's pretty obvious that $g(A) \subseteq U_1$, since $r\cos\theta)^2 + (r\sin\theta)^2 = r^2$, which is between 0 and 1. However, since θ is never equal to 0 or 2π, we never have $y = 0$. Thus, the set $\{(x,y) \in U_1 : y = 0\}$ is not in $g(A)$. However, this set is just a finite line segment in $\mathbb{R}^2$, so is of content 0. Thus, since f is integrable on U_1, it is also integrable on $g(A)$, and:

$$\int_{U_i} f = \int_{g(A)} f$$

Now, note that $g(r,\theta)$ is clearly injective on A, since $\theta \in (0, 2\pi)$ so our sin and cos only oscillate over one period, and r is always positive. So, g is $1-1$ on A. Now, we can use the change of variables theorem to get:

$$\int_{g(A)} f = \int_A f \circ g |\det g'|$$

And,

$$f \circ g = \frac{1}{(r^2\cos^2\theta + r^2\sin^2\theta)} = \frac{1}{r}$$

And as we showed in class, $\det g' = r$. So, we have the integral:

$$\int_A \frac{r}{r^2} = \int_A \frac{1}{r}$$

And then, since this function is continuous on A, we can apply fubini to get:

$$\int_0^1 \int_0^{2\pi} \frac{1}{r} d\theta \, dr = 2\pi \int_0^1 \frac{1}{r} \, dr$$

But this last integral doesn't exist, contradicting our assumption that f is integrable on U_1.

Now, what about U_2? We can do the same thing. Let $A = (1,\infty) \times (0, 2\pi)$. Then, $g(A) = \{(x,y) : x^2 + y^2 > 1, y \neq 0\}$. Again, g is injective on A. This differs from U_2 on an unbounded set. Specifically, it's missing the points on the x axis, specifically the set: $C = \{(x,0) : x^2 > 1\}$. These sets are disjoint, and $U_2 = g(A) \cup C$. So,

$$\int_{U_2} f = \int_{g(A)} f + \int_C f$$

Note that f is integrable on both $g(a)$ and C, since both of these sets have a measure 0 boundary, and the set of discontinities of f is of measure 0, by the assumption that f is integrable on U_2. Then, we consider

$$\int_{g(A)} f$$

Change of variables once again gives us:

$$\int_A f \circ g |\det g'| = \int_A \frac{1}{r} = 2\pi \int_1^\infty \frac{1}{r} dr$$

which again doesn't exist, contradicting our assumption that f is integrable on U_2. So, f is integrable on neither of these sets.

Q3

We have

$$B = \{(x,y) : x > 0, y > 0, 1 < xy < 2, x < y < 4x\}$$

If we let $x = u/v$ and $y = uv$, let's check what the conditions become:

$$x > 0 \Rightarrow u/v > 0 \Rightarrow u > 0$$

$$y > 0 \Rightarrow uv > 0 \Rightarrow v > 0$$

$$1 < xy < 2 \Rightarrow 1 < u^2 < 2 \Rightarrow 1 < u < \sqrt{2}$$

$$x < y < 4x \Rightarrow 1 < y/x < 4 \Rightarrow 1 < v < 4$$

So $B = \{(u/v, uv) \,:\, u \in (1, \sqrt{2}), v \in (1,2)\}$. Now, define $A = \{(u,v) \,:\, u \in (1,\sqrt{2}), v \in (1,2)\}$. Define $g : A \to \mathbb{R}^2$ by $g(u,v) = (\frac{u}{v}, uv)$. We have that $g(A) = B$, essentially by definition. g is also clearly an injection, so it is a bijection from $A \to B$, so we can use change of variables. For that we need $\det g'$, which is given by:

$$g' = \begin{pmatrix} D_1 g_1 & D_2 g_1 \\ D_1 g_2 & D_2 g_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{pmatrix}$$

Which has determinant $\frac{u}{v} + \frac{u}{v} = \frac{2u}{v}$. So, we apply change of variables:

$$\int_B x^2 y^3 = \int_{g(A)} x^2 y^3 \circ g |\det g'| = \int_A u^5 v |\det g'| = \int_A 2u^6$$

$$\begin{aligned} \int_A 2u^6 &= \int_1^{\sqrt{2}} \int_1^2 2u^6 \, dv \, du \\ &= 2 \int_1^{\sqrt{2}} u^6 du \\ &= 2 \left[\frac{u^7}{7} \right]_1^{\sqrt{2}} \\ &= \frac{2}{7}(8\sqrt{2} - 1) \end{aligned}$$

Q4

If we define c to be the open unit tetrahedron, that is, the open tetrahedron with verticies $(0,0,0),(1,0,0),(0,1,0),(0,0,1)$, then if we define a linear map

$$L = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$

Then L sends the verticies of C to the verticies of T, so $Lc = int\, T$. Since the boundary of T is a closed set of measure 0 in $\mathbb{R}^3$, and $\det L = -2$, L is 1-1, we can apply the change of variables theorem.

$$\int_T f = \int_L cf = \int_c f \circ l |\det L| = 2 \int_c f \circ L$$

So,

$$f \circ L = f(L(x,y,z)) = f\left(\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = f(x-z, 2x+y+z, 3x+2y+z)$$

Which is just the function $2x$. So, we want to find:

$$2 \int_c 2x$$

Letting χ_c be the indicator function of c, this is:

$$2 \int_{(0,1)^3} \chi_c \cdot 2x$$

Fubini gives:

$$2 \int_{(0,1)^3} \chi_c \cdot 2x = 2 \int_0^1 \int_0^1 \int_0^1 \chi_c \cdot 2x \, dx \, dy \, dz$$

Now, $\chi_c = 0$ unless $x \in C$. $x, y, z \in c$ if $x, y, z > 0$ and $x + y + z > 1$. So, our integral becomes:

$$\int_0^1 \int_0^1 \int_0^1 \chi_c \cdot 2x \, dx \, dy \, dz = \int_0^1 \int_0^1 \int_0^{1-(y+z)} \chi_c \cdot 2x \, dx \, dy \, dz$$

Since χ_c is 0 if $x \geq 1 - (y+z)$. Then, we also know $\chi_c = 0$ if $y \geq 1 - z$, so our final integral is:

$$2 \int_0^1 \int_0^{1-z} \int_0^{1-(y+z)} 2x \, dx \, dy \, dz$$

since χ_c is now always 1. So, we can integrate:

$$\begin{aligned} 4 \int_0^1 \int_0^{1-z} \int_0^{1-(y+z)} x \, dx \, dy \, dz &= 2 \int_0^1 \int_0^{1-z} y^2 + 2yz - 2y + z^2 - 2z + 1 \, dy \, dz \\ &= -\frac{2}{3} \int_0^1 z^3 - 3z^2 + 3z - 1 \, dz \\ &= \frac{1}{3} \left(-\frac{1}{4} \right) \\ &= \frac{1}{6} \end{aligned}$$

and we're done.

Q5

We want to calculate the integral

$$\int_T 1$$

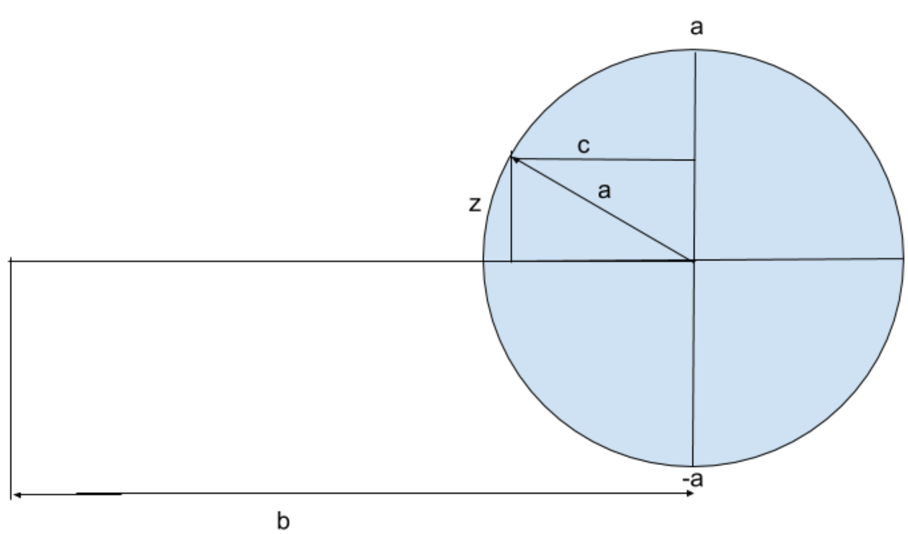
We can use the hint to find a set A with $g(A) = T$. While A is closed, we can just get rid of it's boundary, which has a bounded image of measure 0 (the "surface" of the torus) to have an open set that works for the change of variables theorem. I'll still call this set A. Now, note that g is pretty obviously 1-1. Finally, we need $\det g'$.

$$g' = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This has determinant r. So, change of variables tells us:

$$\int_T 1 = \int_g (A) 1 = \int_A (1 \circ g) \cdot r = \int_A r$$

Now, we want to use fubini, but to do that we need to determine the bounds of the integrals. First of all, our choice of θ has no effect on what r and z need to be, a torus is rotationally symmetric. That means that z can range from the top of the disc of radius A to the bottom, which means from $-a$ to a. Now, if we are given z, what values can r take so that $(r, \theta, z) \in A$?



As we can see from my magnificent diagram, the circle, at height z, goes from $b - \sqrt{a^2 - z^2}$ to $b + \sqrt{a^2 - z^2}$, so that gives us our allowable value of r. So, we have our bounds, and now we can use fubini:

$$\int_A r = \int_0^{2\pi} \int_{-a}^{a} \int_{b-\sqrt{a^2-z^2}}^{b+\sqrt{a^2-z^2}} r \, dr \, dz \, d\theta \tag{1}$$

$$= \int_0^{2\pi} \int_{-a}^{a} 2b\sqrt{a^2 - z^2} \, dz \, d\theta \tag{2}$$

$$= \int_0^{2\pi} \pi a^2 b \, d\theta \tag{3}$$

$$= 2\pi^2 a^2 b \tag{4}$$

And we're done.