Q1

Let $U_1, ...U_n$ be open sets covering the boundary of A (finitely many since the boundary of A has measure 0 and thus content 0), with $B = \bigcup_{i=1}^{n} U_i$, and $V(U_1) + ... + V(U_n) < \varepsilon$. B is a finite union of open sets, and is thus itself open. Now, take $C = A \backslash B$. We know C is closed since C^c is $extA \cup B$, which is a union of open sets and thus open. So, C is closed and contained in A, so it is bounded, meaning that C is compact.

Since C is closed, it contains its boundary. Now, if x is on the boundary of C, then any open ball containing x must contain something in C and something not in C, so it must contain an element of B. Thus, x is also on the boundary of B. So, $Bd(C) \subseteq Bd(B)$. B is a union of open rectangles, so it is integrable. Thus, $Bd(B)$ is of measure 0. So, since $Bd(C) \subseteq Bd(B)$, $Bd(C)$ is also of measure 0. So, C is Jordan Measurable.

Now $V(A \backslash C) = V(A \backslash (A \backslash B)) = V(B) < \varepsilon$.

Q2

Suppose $f : S \subseteq \mathbb{R}^n \to R$.
If there doesn't exist one point a for which $D_{1,2}f(a) - D_{2,1}f(a) \neq 0$, we're done. If there's a point a where $D_{1,2}f(a) - D_{2,1}f(a) \neq 0$, then since $D_{1,2}f(a)$ and $D_{2,1}f(a)$ are continuous, we know that there's an open rectangle A around a with $D_{1,2}f(a) - D_{2,1}f(a) \neq 0$ (and in fact, the difference maintains the same sign) for all $a \in A$, and let $a = (a_1, ...a_n)$.
Let $A = (x_1, x_2) \times (y_1, y_2) \times (b_3, b_4) \times ... \times (b_{2n-1}, b_{2n})$, and define

$$c \subset A = \{(x, y) : (x, y, a_3, ...a_n), x \in (x_1, x_2), y \in (y_1, y_2)\}$$

Then, $c == (x_1, x_2) \times (y_1, y_2)$

Now, consider the function $g(x, y) = f(x, y, a_3, ...a_n)$. We have $G : C \to R$. Also, $D_{1,2}g(x, y) = D_{1,2}f(x, y, a_3, ...a_n)$, and similar for $D_{2,1}g(x, y)$. If $D_{1,2}f(a) - D_{2,1}f(a) \neq 0$ (and difference maintains the same sign) on A, then $D_{1,2}g(x, y) - D_{2,1}g(x, y) \neq 0$ (and the difference maintains the same sign) on c. Thus,

$$\int_c D_{1,2}g(x, y) - D_{2,1}g(x, y) \neq 0$$

Now, both $D_{1,2}g$ and $D_{2,1}g$ are continuous, so we can apply naive fubini.

$$\begin{aligned}
\int_c D_{1,2}g(x, y) - D_{2,1}g(x, y) &= \int_c D_{1,2}g(x, y) - \int_c D_{2,1}g(x, y) \\
&= \int_{(y_1,y_2)} \int_{(x_1,x_2)} \frac{\partial}{\partial x}\frac{\partial}{\partial y} g(x, y)dxdy - \int_{(x_1,x_2)} \int_{(y_1,y_2)} \frac{\partial}{\partial y}\frac{\partial}{\partial x} g(x, y)dydx
\end{aligned}$$

(from naive Fubini)

$$\begin{aligned}
&= \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{\partial}{\partial x}\frac{\partial}{\partial y} g(x, y)dxdy - \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{\partial}{\partial y}\frac{\partial}{\partial x} g(x, y)dydx \\
&= \int_{y_1}^{y_2} \frac{\partial}{\partial y}(g(x_2, y) - g(x_1, y)dy - \int_{x_1}^{x_2} \frac{\partial}{\partial x}(g(x, y_2) - g(x, y_1))dx \\
&= g(x_2, y_2) - g(x_1, y_2) - g(x_2, y_1) + g(x_1, y_1) - [g(x_2, y_2) - g(x_2, y_1) - g(x_1, y_2) + g(x_1, y_1)] \\
&= g(x_2, y_2) - g(x_1, y_2) - g(x_2, y_1) + g(x_1, y_1) - g(x_2, y_2) + g(x_2, y_1) + g(x_1, y_2) - g(x_1, y_1) \\
&= 0
\end{aligned}$$

So, the integral is 0, meaning that no a exists such that $D_{1,2}f(a) - D_{2,1}f(a) \neq 0$. So, they're equal and we're done.

Q3

Define $C \subseteq [a,b] \times [c,d]$ by $C = [a,b] \times [c,y]$ for some constant y. Then,

$$F(y) = \int_a^b f(x,y)dx = \int_a^b f(x,y) - f(x,c) + f(x,c)dx = \int_a^b \int_c^y D_2 f(x,y)dy + f(x,c)dx = \int_C D_2 f(x,y) + \int_a^b f(x,c)dx$$

Then, we have:

$$\begin{aligned}
F'(y) &= \frac{d}{dy}(\int_C D_2 f(x,y) + \int_a^b f(x,c)dx) \\
&= \frac{d}{dy}\int_C f(x,y) \\
&= \frac{d}{dy}\int_c^y \int_a^b D_2 f(x,y)dxdy \\
&= \int_a^b D_2 f(x,y)dx
\end{aligned}$$

As desired, so we're done. Note that if we had $F(x) = \int_c^d f(x,y)dy$ we could have applied the same argument to show that $F'(x) = \int_c^d D_1 f(x,y)dy$.

Q4

First, consider

$$D_1 f(x,y) = D_1 \int_0^x g_1(t,0)dt + D_1 \int_0^y g_2(x,t)dt$$

So, let's find $D_1 \int_0^x g_1(t,0)dt$ first. This is just equal to $g_1(x,0)$. Now, for $D_1 \int_0^y g_2(x,t)dt$, we can set $h(x) = \int_0^y g_2(x,t)dt$. Then, applying question 3, we get

$$h'(x) = \int_0^y D_1 g_2(x,t)dt = \int_0^y D_2 g_1(x,t)dt = g_1(x,0) - g_1(x,y)$$

So,

$$D_1 f(x,y) = D_1 \int_0^x g_1(t,0)dt + D_1 \int_0^y g_2(x,t)dt = g_1(x,0) + g_1(x,y) - g_1(x,0) = g_1(x,y)$$

So that's done. Now, let's consider

$$D_2 f(x,y) = D_2 \int_0^x g_1(t,0)dt + D_2 \int_0^y g_2(x,t)dt$$

First, note that $g_1(t,0) = D_1 f(t,0)$, so we have:

$$D_2 \int_0^x D_1 f(t,0)dt = D_2(f(x,0) - f(0,0)) = 0$$

Now, $D_2 \int_0^y g_2(x,t)dt$ is just $g_2(x,y)$. So, putting it all together, we get:

$$D_2 f(x,y) = D_2 \int_0^x g_1(t,0)dt + D_2 \int_0^y g_2(x,t)dt = 0 + g_2(x,y) = g_2(x,y)$$

So, we're done.

Q5

a) First, note that if A is bounded, so is $L(A)$, since for each $x \in A$, $|L(x)| \leq C|x|$ for a specific c determined by the linear map.
Since A is Jordan Measurable, we know that χ_A is discontinuous only on a set of measure 0. Now, we know that each of the linear transformations provided are invertible, meaning they are bijections. Also, they're continuous (since linear transformations are continuous). Now, consider the function $\chi = \chi_A \circ L^{-1}$. If $x \in L(A)$, $\chi(x) = \chi_A L^{-1}(x)$. Since $L^{-1}x \in A$, $\chi_A L^{-1}(x) = 1$. Similarly, if $x \notin L(A), L^{-1}(x) \notin A$, so $\chi(x) = \chi_A L^{-1}(x) = 0$. So, we have that $\chi = \chi_{L(A)}$.

So, what are the discontinuities of $\chi_L(A) \circ L^{-1}$? First of all, note that since the set of discontinuities of χ_A is closed and of measure 0, so the set of discontinuities has content 0, we can find a list of rectangles $U_1, ...U_k$ which cover the set of discontinuities of χ_A, and whose sum of volumes is less than ε for any given epsilon. Now, any rectangle is the union of a finite number of n-cubes, so we can assume wlog that the $U_1, ...U_k$ are n-cubes, each with side length l_i. So, the sum of the volumes of these cubes is $l_1^n + ... + l_k^n$.

Now:

Lemma 1. *If B is a ball $\{x : |x - a| \leq d\}$, and L is a bijection, then $L(B) \subseteq \{x : |x - La| \leq Cd\}$.*

Proof. We know that any linear operator L satisfies $|Lx| \leq C|x|$ for some C. Now, suppose $x \in B$. Then, $|Lx - La| = |L(x - a)| \leq C|x - a| \leq Cd$, so $|Lx - La| \leq Cd$, So $L(B) \subseteq \{x : |x - La| \leq Cd\}$. □

Let ε be given. We want to find a set of rectangles covering the discontinuities of $\chi_A \circ L^{-1}$ whose sum of volumes is less than ε.

There exist $U_1, ...U_k$ covering the discontinuities of χ_A such that $l_i^n + ... + l_k^n < \frac{\varepsilon}{2^n C^n}$.

Now, consider balls containing $U_1, ...U_k$. Let $U_i \subset B_i$, with B_i being a ball of radius $2l_i$. Now, $L(U_i) \subset L(B_i) \subset B_i'$, where B_i' is a ball of radius $2Cl_i$ (from the lemma). Note that since the U_ia cover the set of discontinuities of χ_A, and L^{-1} is a continuous bijection, then each discontinuity of $\chi_A \circ L^{-1}$ is in one of the LU_is, and therefore B_i'.

Each B_i' is contained in a cube of side length $2Cl_i$, so the cubes $C_i, ...C_k$ cover the set of discontinuities of $\chi_A \circ L^{-1}$. Then, the sum of the volumes of these cubes is $(2Cl_1)^n + ...(2Cl_k)^n = 2^n C^n (l_1^n + ...l_k^n)$. However, we have picked our $U_1, ...U_k$ so that $l_i^n + ... + l_k^n < \frac{\varepsilon}{2^n C^n}$, so we get:

$$\sum_{i=1}^{k} V(C_i) = 2^n C^n (l_1^n + ...l_k^n) < 2^n C^n \frac{\varepsilon}{2^n C^n} = \varepsilon$$

So, $C_1, ...C_k$ cover the set of discontinuities of $\chi_A \circ L^{-1}$, and their volumes sum to less that ε, so the set of discontinuities of $\chi_A \circ L^{-1}$ is of measure 0.

But, we established that $\chi_A \circ L^{-1} = \chi_{L(A)}$, meaning that the set of discontinuities of $\chi_{L(A)}$ is of measure 0, so $L(A)$ is Jordan measurable.

b)

Case 1: Consider the first type of operator, of the form

$$L(e_k) = \begin{cases} \alpha e_i & k = i \\ e_k & else \end{cases}$$

Note that $\det L = \alpha$.

Lemma 2. *If A is a rectangle of the form $[a_1, a_2] \times ... \times [a_i, b_i] \times ...[a_n, b_n]$, then $V(A) = |\alpha|V(A)$.*

Proof. First, suppose $\alpha > 0$ Consider the rectangle $B = [a_1, a_2] \times ... \times [\alpha a_i, \alpha b_i] \times ...[a_n, b_n]$. I claim $L(A) = B$. First, suppose $x \in A$. Then, letting $a_c \in [a_c, b_c]$, $x = (a_1, ...a_i, ...a_n)$. Then, $L(x) = (a_1, ...\alpha a_i, ...a_n) \in B$.

Now, suppose $x \in B$. Then, $x = (a_1, ...\alpha a_i, ...a_n) = L(a_1, ...a_i, ...a_n) \in L(A)$, so $x \in L(A)$. So, $L(A) = B$. Then, $V(L(A)) = V(B) = (b_1-a_1)...\alpha(b_i-a_i)...(b_n-a_n) = \alpha(b_1-a_1)...(b_i-a_i)...(b_n-a_n) = \alpha V(A) = |\det L|V(A)$.

Now, consider $\alpha < 0$. If we define $B = [a_1, a_2] \times ... \times [\alpha b_i, \alpha a_i] \times ...[a_n, b_n]$, a similar argument to before shows that $L(A) = B$. Then, $V(L(A)) = V(B) = (b_1 - a_1)...\alpha(a_i - b_i)...(b_n - a_n) = (b_1 - a_1)...(-\alpha)(b_i - a_i)...(b_n - a_n) = (-\alpha)(b_1 - a_1)...(b_i - a_i)...(b_n - a_n) = -\alpha V(A) = |\det L|V(A)$. $\square$

Case 2: Consider the second type of operator. If $i < j$, then $detL = -1$. If $i > j$, $\det L = 1$. In both cases, $|\det L| = 1$.

$$L(e_k) = \begin{cases} e_j & k = i \\ e_i & k = j \\ e_k & else \end{cases}$$

Lemma 3. *If A is a rectangle of the form $[a_1, a_2] \times ... \times [a_i, b_i] \times ... \times [a_j, b_j] \times ...[a_n, b_n]$, then $V(A) = V(L(A))$. I'm proving this for $i < j$ since the $j > i$ case is nearly identical.*

Proof. First, consider the rectangle $B = [a_1, a_2] \times ... \times [a_j, b_j] \times ... \times [a_i, b_i] \times ...[a_n, b_n]$. I claim that $L(A) = B$. First, suppose $x \in A$. Then, letting $a_c \in [a_c, b_c]$, $x = (a_1, ...a_i, ...a_j, ...a_n)$. $L(x) = (a_1, ...a_j, ...a_i, ...a_n) \in B$. Now, suppose $x \in B$. Then, $x = (a_1, ...a_j, ...a_i, ...a_n)$. But then $x = L(a_1, ...a_i, ...a_j, ...a_n)$. So, $x \in L(A)$. Thus, $B = L(A)$. Now, $V(L(A)) = V(B) = (b_1 - a_1)...(b_j - a_j)...(b_i - a_i)...(b_n - a_n) = 1 \cdot V(A) = |\det L| \cdot V(A)$. $\square$

Case 3: Consider the third type of operator. Wlog, we can let L be an operator of the form

$$L(e_k) = \begin{cases} e_2 + e_1 & k = 1 \\ e_k & else \end{cases}$$

Since if it isn't, it can be made into one by applying certain operators of form (2) to A, which we already know don't change the volume of A.

Lemma 4. *If A is a rectangle of the form $[a_1, a_2] \times ... \times [a_n, b_n]$, and $R = [c_1, d_1] \times ... \times [c_n, d_n]$ is a rectangle containing both A and $L(A)$, then $V(L(A)) = |det(L)|V(A) = V(A)$ (since $det(L) = 1$).*

Proof. We know that $V(L(A)) = \int_R \chi_{L(A)} = \int_R \chi_A L^{-1}$. Then, consider the function $\chi_{i,(a_1,...a_{n-1})}(x) : \mathbb{R} \to \mathbb{R}$ defined by $\chi_{i,(a_1,...a_{n-1})}(x) = \chi_A L^{-1}(a_1, ...x_i, ...a_{n-1})$. For any i and $(a_1, ...x_i, ...a_n)$, this function is discontinuous at at most two points, the points where the line $\{(a_1, ...x_i, ...a_n) : x_i \in \mathbb{R}\}$ intersects $L(A)$.

Now, apply fubini's theorem:

$$\int_R \chi_{L(A)} = \int_{c_n}^{d_n} ... \int_{c_1}^{d_1} \chi_A L^{-1}(x_1, ...x_n) dx_1...dx_n$$

We can use naive fubini here because at every step of the iterated integral we're integrating $\chi_{i,(a_1,...a_{n-1})}(x)$ for some i and $(a_1,...a_{n-1})$, which is continuous except at maybe two points. Now, we can rearrange the integrals:

$$= \int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_{c_n}^{d^n}...\int_{c_3}^{d_3}\chi_A L^{-1}(x_1,...x_n)dx_3...dx_n dx_2 dx_1$$

Now, note that $L^{-1}(x_1,...x_n) = (x_1, x_2 - x_1, ...x_n)$. So, we have:

$$= \int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_{c_n}^{d^n}...\int_{c_3}^{d_3}\chi_A(x_1,x_2-x_1,...x_n)dx_3...dx_n dx_2 dx_1 \quad (1)$$

$$= \int_{c_1}^{d_1}\int_{c_2}^{d_2}\left(\int_{c_n}^{d^n}...\int_{c_3}^{d_3}\chi_A(x_1,x_2-x_1,...x_n)dx_3...dx_n\right)dx_2 dx_1 \quad (2)$$

Now, $\chi_A(x_1,x_2-x_1,...x_n)$ is always 0 unless $x_3 \in [a_3,b_3],...,x_n \in [a_n,b_n]$, so our inner integral becomes:

$$\int_{c_n}^{d^n}...\int_{c_3}^{d_3}\chi_A(x_1,x_2-x_1,...x_n)dx_3...dx_n = \int_{a_n}^{b_n}...\int_{a_3}^{b_3}\chi_A(x_1,x_2-x_1,...x_n)dx_3,...dx_n$$

Now, the inner function is 1 if $x_1 \in [a_1,b_1]$, and if $x_2 - x_1 \in [a_2,b_2]$, and 0 otherwise. If the inner function is equal to 1, the integral becomes:

$$\int_{a_n}^{b_n}...\int_{a_3}^{b_3}\chi_A(x_1,x_2-x_1,...x_n)dx_3,...dx_n = \int_{a_n}^{b_n}...\int_{a_3}^{b_3}1dx_3,...dx_n = (b_3-a_3)(b_4-a_4)...(b_n-a_n)$$

So, we can define a new function, $X : \mathbb{R}^2 \to \mathbb{R}$ by

$$X(x_1,x_2) = \begin{cases}(b_3-a_3)(b_4-a_4)...(b_n-a_n) & x_1 \in [a_1,b_1] \ and \ x_2 - x{\in}[a_2,b_2] \\ 0 & else\end{cases}$$

and we're left with

$$\int_R \chi_{L(A)} = \int_{c_1}^{d_1}\int_{c_2}^{d_2}X(x_1,x_2)dx_2dx_1$$

But this function is 0 if $x_2 \notin [a_2+x_1, b_2+x_1]$, and $x_1 \notin [a_1,b_1]$, so the integral is equal to:

$$\int_{a_1}^{b_1}\int_{a_2+x_1}^{b_2+x_1}X(x_1,x_2)dx_2dx_1$$

And the function $X(x_1,x_2)$ is constant and equal to $(b_3-a_3)(b_4-a_4)...(b_n-a_n)$ on those bounds, so we finally get:

$$\int_{a_1}^{b_1}\int_{a_2+x_1}^{b_2+x_1}X(x_1,x_2)dx_2dx_1 = \int_{a_1}^{b_1}(b_2+x_1-a_2-x_1)(b_3-a_3)...(b_n-a_n)dx_1 = (b_1-a_1)(b_2-a_2)...(b_n-a_n)$$

Which is equal to $V(A)$, so we're done. $\square$

Now, let R be a cube containing $A, L(A)$. Let P be a partition of R into cubes. Then, let $c_1,...c_n \in P$ be cubes such that $c_i \subseteq A$. Let $d_1,...d_m \in P$ be cubes such that $d_i \cap A \neq \emptyset$.

Consider $U(\chi_A, P) = \sum_{S\in P}\sup_S \chi_A V(s)$. Since $\sup_S \chi_A \neq 0 \iff S \cap A \neq \emptyset$, we have that $U(\chi_A,P) = \sum_{i=1}^m V(d_i)$. Similarly, $L(\chi_A,P) = \sum_{S\in P}\inf_S \chi_A V(s)$, and a term of this sum is nonzero iff $S \subseteq A$. So, again, the only nonzero terms are the ones corresponding to the c_is, so $L(\chi_A,P) = \sum_{i=1}^n V(c_i)$.

Now, it's also true that

$$\bigcup_{i=1}^{n} c_i \subseteq A \subseteq \bigcup_{i=1}^{m} d_i$$

Then, we can apply one of the functions discussed above, and since they're all bijections, we get:

$$\bigcup_{i=1}^{n} Lc_i \subseteq L(A) \subseteq \bigcup_{i=1}^{m} Ld_i$$

Taking volumes, we get:

$$\sum_{i=1}^{n} V(Lc_i) \leq V(L(A)) \leq \sum_{i=1}^{m} V(Ld_i)$$

Since c_i and d_i are cubes, and therefore rectangles, we can apply the above lemmas to get:

$$|\det L| \sum_{i=1}^{n} V(c_i) \leq V(L(A)) \leq |\det A| \sum_{i=1}^{m} V(d_i)$$

and, using the identities from above:

$$|\det L| L(\chi_A, P) \leq V(L(A)) \leq |\det A| U(\chi_A, P) V(d_i)$$

And since the supremum of the left side is the infimum of the right side, due to A's jordan measurably, we find that $|\det L| \int_R \chi_A = |\det L| V(A) = V(L(A))$, as desired.

c)

Now, consider some arbitrary linear map, L. If $\det L = 0$, then it sends A to a lower dimensional subspace of $\mathbb{R}^n$, meaning that $L(A)$ is of measure 0 in $\mathbb{R}^n$, so $V(L(A)) = 0$. So, suppose $\det L \neq 0$. Then row reduction takes L to the identity. However, row reduction is equivalent to multiplying L by the three types of matricies given above, in some form. So, some product of the above types of matricies take L to the identity, meaning that this product is the inverse of L. Suppose $L^{-1} = A_1 ... A_n$. Then, $L = A_n^{-1} ... A_1^{-1}$. All A_n^{-1} are of the same class of matrix as A_n was. Thus, L is a product of matrices of the above type.

Now, let n be the minimum number of matricies of the above type that multiply together to make L. We want to show by induction on n that $V(L(A)) = |\det L| V(A)$. For our base case, take $n = 1$. Then, L is one of the above types of matricies, and we're done.

Now, suppose for matricies L that are equal to the product of n matricies, $V(L(A)) = |\det L| V(A)$. Suppose L is the product of $n+1$ matricies. Then, $L = MN$, where N is the product of n matricies and M is a matrix of the above type. Then, $V(L(A)) = V(MN(A)) = V(M(N(A)))$.
Let $B = N(A)$ $V(B) = |\det N| V(A)$ by the inductive hypothesis. Then, $V(M(N(A))) = V(M(B)) = |\det M| V(B) = |\det M| |\det N| V(A) = |\det(MN)| V(A) = |\det L| V(A)$, so we're done.