Q1

a) We know that $f \geq \inf(f)$ and $g \geq \inf(g)$. Thus, $f + g \geq \inf(f) + \inf(g)$, meaning that $\inf(f) + \inf(g)$ is a lower bound for $f + g$. Since $\inf(f + g)$ is the *greatest* lower bound for $f + g$, we have that $\inf(f + g) \geq \inf(f) + \inf(g)$. If we consider restrictions of f, g and thus $f + g$ to some set A, this still holds true. In particular, restricting all functions to S, we have $m_s(f) + m_s(g) \leq m_s(f + g)$.

Similarly, $f \leq \sup(f)$, and $g \leq \sup(g)$, so $f + g \leq \sup(f) + \sup(g) \implies \sup(f + g) \leq \sup(f) + \sup(g)$, and again, this holds true provided we consider all supremums on the same domain, so $M_s(f + g) \leq M_s(f) + M_s(g)$.

Then,

$$\begin{aligned} L(f, P) + L(g, P) &= \sum_{s \in P} m_s(f)v(s) + \sum_{s \in P} m_s(g)v_(s) \\ &= \sum_{s \in P} v(s)(m_s(f) + m_s(g)) \\ &\leq \sum_{s \in P} v(s)m_s(f + g) \\ &= L(f + g, P) \end{aligned}$$

So,

$$L(f, P) + L(g, P) \leq L(f + g, P)$$

Similarly,

$$\begin{aligned} U(f, P) + G(f, P) &= \sum_{s \in P} M_s(f)v(s) + \sum_{s \in P} M_s(g)v(s) \\ \\ &= \sum_{s \in P} v(s)(M_s(f) + M_s(g)) \\ &\geq \sum_{s \in P} v(s)M_s(f + g) \\ &= L(f + g, P) \end{aligned}$$

Thus,

$$U(f, P) + U(g, P) \geq U(f + g, P)$$

b) Let ε be given. We want to show that there exists a partition P with $U(f + g, P) - L(f + g, P) < \varepsilon$. We know that both f and g are integrable, so we can pick partitions P_1 and P_2 with

$$\begin{aligned} U(f, P_1) - L(f, P_1) &< \frac{\varepsilon}{2} \\ U(g, P_2) - L(g, P_2) &< \frac{\varepsilon}{2} \end{aligned}$$

And if we pick a P refining P_1 and P_2, say, for simplicity, $P_1 \cap P_2$, we find:

$$\begin{aligned} U(f, P) - L(f, P) &< \frac{\varepsilon}{2} \\ U(g, P) - L(g, P) &< \frac{\varepsilon}{2} \end{aligned}$$

Now, we can just add these two inequalities to get:

$$U(f,P) + U(g,P) - (L(f,P) + L(g,P)) < \varepsilon$$

But

$$U(f,P) + U(g,P) - (L(f,P) + L(g,P)) \geq U(f+g,P) - L(f+g,P)$$

So, $U(f+g,P) - L(f+g,P) < \varepsilon$, so $f+g$ is integrable.

Now, we want to find $\inf(U(f+g,P)$. (We could also find $\sup(L(f+g,P))$, since we know that they're equal), Since $f+g$ is integrable, this value is equal to the value $\int_A f+g$.

Now, recall that since f,g are integrable, $\int_A f + \int_A g = \sup(L(f,P)) + \sup(L(g,P))$. So, note that

$$U(f,P) + U(g,P) \geq U(f+g,P) \geq L(f+g,P) \geq L(f,P) + L(g,P)$$

So,

$$U(f,P) + U(g,P) \geq L(f+g,P) \geq L(f,P) + L(g,P)$$

Thus,

$$U(f,P) - L(f,P) + U(g,P) - L(g,P) \geq L(f+g,P) - (L(f,P) + L(g,P))$$

So, for all $\varepsilon > 0$, there exits a P where

$$\varepsilon > L(f+g,P) - (L(f,P) + L(g,P)) \geq L(f+g,P) - (\sup(L(f,P)) + \sup(L(g,P))) \geq 0$$

Specifically,

$$\varepsilon > L(f+g,P) - (L(f) + L(g)) \geq 0$$

(Using $L(f) = \sup(L(f,P))$ and similar for g)

Which means that $L(f+g) = L(f) + L(g)$, meaning $\int_A f+g = L(f+g) = L(f) + L(g) = \int_A f + \int_A g$.

c)

First of all, if $c = 0$, then $cf = 0$, which is obviously integrable with integral 0. So, for the rest of this, let $c \neq 0$.

If f is integrable, then for any given partition P,

$$U(cf,P) = \sum_{s\in P} M_s(f)v(s) = \sum_{s\in P} \sup(cf)v(s) = \sum_{s\in P} c\sup(f)v(s) = c\sum_{s\in P} \sup(f)v(s) = cU(f,P)$$

Similarly, $L(cf,P) = cL(f,P)$. Let's first show that cf is integrable.

Let ε be given. Then, $U(cf,P) - L(cf,P) = c(U(f,P) - L(f,P)$. Since f is integrable, there exists a partition P where $U(f,P) - L(f,P) < \frac{\varepsilon}{c}$. Then, using that P, we have $U(cf,P) - L(cf,P) = c(U(f,P) - L(f,P)) < c\frac{\varepsilon}{c} = \varepsilon$, so cf is integrable.

We want to find $\sup(L(cf,P))$. Note that this is equal to $c\sup(L(f,P))$. But, we know that $\sup(L(f,P)) = \int_A f$, so $\sup(L(cf,P)) = c\int_A f$, as desired.

Q2

First, assume f is integrable on A. That means that for all $\varepsilon > 0$ $\exists P_1$ with $U(f, P_1) - L(f, P_1) < \varepsilon$. In order to show that $F|_s$ is integrable, we want to show that there exists a partition, P_s, of s, with $U(F|_s, P_s) - L(F|_s, P_s) < \varepsilon$.

Consider the partition $P_2 = P \cup P_1$. Then, we know that P_2 contains the endpoints of each $s \in P$. Furthermore, it is the union of nearly disjoint partitions of each $s \in P$. Specifically, $P_2 = \cup_{s \in P} P_s$, where P_s is a partitioning of s.

That means that $U(f, P_2) = \sum_{s' \in P_2} M_{s'}(f)v(s') = \sum_{s \in P} \sum_{s' \in P_s} M_{s'}(f)v(s')$, since each $s' \in P_2$ is also in some P_s. Then, since on a given rectangle S, each $f = F|_S$, we can write:

$$U(f, P_2) = \sum_{s \in P} \sum_{s' \in P_s} M_{s'}(F|_S)v(s') = \sum_{s \in P} U(F|_S, P_s)$$

By a similar argument, $L(f, P_2) = \sum_{s \in P} \sum_{s' \in P_s} M_{s'}(f)v(s')$ So,

$$L(f, P_2) = \sum_{s \in P} \sum_{s' \in P_s} m_{s'}(F|_S)v(s') = \sum_{s \in P} L(F|_S, P_s)$$

But, since P_2 is a refinement of P_1, we know that

$$U(f, P_2) - L(f, P_2) \leq U(f, P_1) - L(f, P_1) < \varepsilon$$

So, $U(f, P_2) - L(f, P_2) < \varepsilon$. But, expanding the sums, we get:

$$U(f, P_2) - L(f, P_2) = \sum_{s \in P} U(F|_S, P_s) - \sum_{s \in P} L(F|_S, P_s) = \sum_{s \in P} U(F|_S, P_s) - L(F|_S, P_s) < \varepsilon$$

But, each term of the rightmost sum is nonnegative, so each term of the sum must be less than ε. Thus, $F|_S$ is integrable on each $s \in P$.

Now, note that $\int_A f = L(f)$. We know $L(f) = \sup L(f, P_3)$. Now, wlog, assume $P \subseteq P_3$. Then, we know $L(f) = \sup(\sum_{s \in P} \sum_{s' \in S_p} m_{s'}(f)v(s)) = \sup \sum_{s \in P} L(F|_S, P_s)$. However, as we refine the partition P_3, which P_s is a subset of, we also refine the P_s's. Thus, as P_3 becomes arbitrarily refined, and $\sum_{s \in P} L(F|_S, P_s)$ approaches its supremum, so does each term of the sum. Thus, $\sup \sum_{s \in P} L(F|_S, P_s) = \sum_{s \in P} \sup L(F|_S, P_s)$. But, since we already know that $F|_S$ is integrable on each s, then we have $\int_A f = L(f) = \sum_{s \in P} \int_s F|_S$, as desired.

Now, for the other direction, assume each $F|_S$ integrable on s. Then, let k be the number of rectangles in P_s. Let ε be given. We know that there exists a P_s for each s where

$$U(F|_S, P_s) - L(F|_S, P_s) < \frac{\varepsilon}{k}$$

Then, letting $P_1 = \cup_{s \in P} P_s$, we have:

$$U(f, P_1) = \sum_{s_1 \in P_1} M_s(f)v(s) = \sum_{s \in P} \sum_{s' \in P_s} M_{s'}v(s') = \sum_{s \in P} U(F|_S, P_s)$$

and similarly,

$$L(f, P_1) = \sum_{s \in P} L(F|s, P_s)$$

So,

$$U(f, P_1) - L(f, P_1) = \sum_{s \in P} U(F|_S, P_s) - L(F|_S, P_s) < \sum_{s \in P} \frac{\varepsilon}{k} = k\frac{\varepsilon}{k} = \varepsilon$$

So, f is integrable on A.

To find the integral of f, we'll use essentially the same argument as for the first direction. Wlog, take a partition P_2 where $P \subseteq P_2$. Then, find the supremum of $L(f, P_2)$ under refinements of P_2:

$$L(f) = \sup L(f, P_2) = \sup \sum_{s \in P} L(F|_S, P_s) = \sum_{s \in P} \sup L(F|_S, P_s) = \sum_{s \in P} \int_s F|_S$$

And we're done.

☺

Q3

Since f and g are integrable, we know that $L(f) = U(f) = \int_A f$, and similarly for g. Now, we have:

$$U(f) = \sup \sum_{s \in P} M_s(f)v(s) \tag{1}$$

Now, we know that $M_s(f)$ is the supremum of f on the rectangle s. Since $f \leq g$, we know that on all rectangles s, that $\sup f \leq \sup g$. Thus, $M_s(g) \geq M_s(f)$. Then, we have:

$$\sum_{s \in P} M_s(f)v(s) \leq \sum_{s \in P} M_s(g)v(s) \implies \sup \sum_{s \in P} M_s(f)v(s) \leq \sup \sum_{s \in P} M_s(g)v(s) \tag{2}$$

So, we get:

$$\int_A f = \sup \sum_{s \in P} M_s(f)v(s) \leq \sup \sum_{s \in P} M_s(g)v(s) = \int_A g$$

So, $\int_A f \leq \int_A g$, as desired.

Q4

Let s be some rectangle with a nonempty intersection with A. I want to show that $M_s(|f|) - m_s(|f|) \leq M_s(f) - m_s(f)$. We have a couple of cases. First, if $M_s(f) - m_s(f)$ have the same sign, then $M_s(|f|) - m_s(|f|) = M_s(f) - m_s(f)$. If they have different signs, then in particular, $m_s(f) < 0$, and $M_s(f) \geq 0$. Then, note that $M_s(|f|) = |m_s(f)|$.

In that case,

$$M_s(|f|) - m_s(|f|) \leq M_s(|f|) = |m_s(f) - 0| \leq |m_s(f) - M_s(f)|$$

Where the last inequality comes from the fact that $M_s(f) \geq 0$. So, we have:

$$M_s(|f|) - m_s(|f|) \leq |m_s(f) - M_s(f) = M_s(f) - m_s(f)$$

So, in both cases, we have

$$M_s(|f|) - m_s(|f|) \leq M_s(f) - m_s(f)$$

As desired.

Now, we want to show that $|f|$ is integrable. Note that $|f|$ is bounded, since f is bounded. Then, we want to show that for all $\varepsilon > 0$ there exists a P such that $U(|f|, P) - L(|f|, P) < \varepsilon$. So, let ε be given.

Then, we know since f is integrable, we can find a p such that

$$U(f, P) - L(f, P) < \varepsilon$$

But:

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{s \in P} v(s)(M_s(f) - m_s(f)) \\ &\geq \sum_{s \in P} v(s)(M_s(|f|) - m_s(|f|)) \\ &= U(|f|, P) - L(|f|, P) \end{aligned}$$

So, we have

$$\varepsilon > U(f, P) - L(f, P) \geq U(|f|, P) - L(|f|, P)$$

Meaning

$$\varepsilon > U(|f|, P) - L(|f|, P)$$

as desired. So, $|f|$ is integrable.

Now, note that since f is integrable, $\inf U(f, P) = U(f) = \int_A f$. Then, we have two cases. First, suppose that $U(f, P) \geq 0$. Then, note that:

$$\begin{aligned} U(f, P) &= \sum_{s \in P} M_s(f) v(s) \\ &\leq \sum_{s \in P} M_s(|f|) v(s) \\ &= U(|f|, P) \end{aligned}$$

With the last inequality coming from the fact that $M_s(f) \leq M_s(|f|)$. So,

$$U(f, P) \leq U(|f|, P) \implies U(f) \leq U(|f|)$$

And, since we're assuming that $U(f) \geq 0$, we have:

$$\left| \int_A (f) \right| = |U(f)| = U(f) \leq U(|f|) = \int_A |f|$$

And we're done.

Now, assume that $U(f) \leq 0$. Then, we know that there's some partition P where $U(f,P) < 0$. In that case, a lower bound for $U(f,P)$ is an upper bound for $|U(f,P)|$, meaning that $|U(f)| = |\inf U(f,P)| = \sup |U(f,P)|$. Now:

$$\begin{aligned} |U(f,P)| &= \left| \sum_{s \in P} M_s(f) v(s) \right| \\ &\leq \sum_{s \in P} |M_s(f)| v(s) \\ &\leq \sum_{s \in P} M_s(|f|) v(s) \\ &= U(|f|, P) \end{aligned}$$

So, we have $|U(f,P)| \leq U(|f|,P)$. Therefore, $\sup |U(f,P)| \leq \inf U(|f|,P)$. Then:

$$\left| \int_A f \right| = |\inf U(f,P)| = \sup |U(f,P)| \leq \inf U(|f|,P) = U(|f|) = \int_A |f|$$

Which is what we wanted, so we've covered both cases, and are therefore done.

Q5

a)

Let A be some unbounded set, and let $\{U_1, ...U_n\}$ be a list of closed rectangles. In particular, we know each U_i is bounded. Then,

$$\bigcup_{i=1}^{n} U_i$$

is bounded, since it's a finite union of bounded sets. So, it can't cover A, since A is not bounded. Thus, no finite list of open rectangles covers A, so A does not have content 0.

b)

Consider the x axis in $\mathbb{R}^2$. It has measure 0, since for all $\varepsilon > 0$ we can cover it with closed rectangles of the form $U_k = [-\frac{k\varepsilon}{4}, \frac{k\varepsilon}{4}] \times [-\frac{1}{k2^{k+1}}, \frac{1}{k2^{k+1}}]$. Then, any point of the form $(a, 0) \in \mathbb{R}^2$ is clearly in one of the U_k, but the volume of each U_k is:

$$v(U_k) = \frac{k\varepsilon}{2} \frac{1}{k2^k} = \frac{\varepsilon}{2} \frac{1}{2^k}$$

Meaning that $\sum_{k=1}^{\infty} U_k = \frac{\varepsilon}{2} \left(\sum_{k=1}^{\infty} \frac{1}{2^k}\right) = \frac{\varepsilon}{2}$.
So, the x axis is of measure 0 in $\mathbb{R}^2$, but it is unbounded, and therefore does not have content 0.

Q6

First, consider the set $A' = A \cap [0,1]$. Then, $A' \subseteq [0,1]$, and we know from pset 1 q 6 that $bd(A') = [0,1] - A' = [0,1] - A$.

Furthermore, if a point in $(0,1)$ is on the boundary of A', it is obviously also on the boundary of A. However, $0, 1$ are always on the boundary of A', and may not be on the boundary of A (it in fact might be the case that they always are on the boundary of A, but it actually doesn't matter). Thus, $bd(A') = bd(A) \cap (0,1) \cup \{0,1\} = [0,1] - A$.

However, if $[0,1] - A$ is not of measure 0, then we know that $bd(A) \cap (0,1)$ must not be of measure 0, since if it was we'd have $bd(A) \cap (0,1) \cup \{0,1\}$ being the union of two measure 0 sets, which would be of measure 0. If $bd(A) \cap (0,1)$ is not of measure 0, neither is $bd(A)$, since $bd(A) \cap (0,1) \subseteq A$. Thus, $[0,1] - A$ not being of measure 0 implies that $bd(A)$ is not of measure 0.

Now, note that since $A = \bigcup_{i-1}^{\infty}(a_i, b_i)$, we know that $\{[a_i, b_i]\}$ covers A. Suppose for the sake of contradiction that the set $[0,1] - A$ is of measure 0. Then, we can cover it with closed rectangles U_j, with the property that for all ε there exists a set of closed rectangles U_j with $\sum_{i=1}^{\infty} v(U_i) < \varepsilon$.

Now, let $\{U_j\}$ be some arbitrary set of rectangles covering $[0,1] - A$. Then, $\{U_j\}$ covers $[0,1] - A$, and $\{[a_i, b_i]\}$ covers A, so the union of these two sets covers $[0,1]$. So, $\sum_{j=1}^{\infty} v(U_j) + \sum_{i=1}^{\infty} v([a_i, b_i]) \geq v([0,1]) = 1$. Thus:

$$\sum_{i=1}^{\infty} v([a_i, b_i]) \geq 1 - \sum_{j=1}^{\infty} v(U_j)$$

And this is true for all sets $\{U_i\}$ that cover $[0,1] - A$. So, for any number $a < 1$, we can set $\varepsilon = 1 - a$ and find a cover of $[0,1] - A$ with $\sum_{j=1}^{\infty} v(U_j) < \varepsilon$. But then:

$$\begin{aligned}\sum_{i=1}^{\infty} v([a_i, b_i]) &\geq 1 - \sum_{j=1}^{\infty} v(U_j) \\ &> 1 - \varepsilon \\ &= 1 - (1 - a) \\ &= a\end{aligned}$$

Meaning for all $a < 1$ we have:

$$\sum_{i=1}^{\infty} v([a_i, b_i]) > a$$

So, $\sum_{i=1}^{\infty} v([a_i, b_i]) \geq 1$. But:

$$\sum_{i=1}^{\infty} v([a_i, b_i]) = \sum_{i=1}^{\infty} b_i - a_i > 1$$

Which we know to be not the case, so $[0,1] - A$ cannot be of measure 0, and thus neither can $bd(A)$, so we're done.

Q7

First, consider $o(f,x)$. We have defined $o(f,x) = \lim_{r\to 0} o(f, B_r(x)) = \lim_{r\to 0} \sup_{B_r(x)}(f) - \inf_{B_r(x)}(f)$. Now, consider $x_1 > x$. Then, we we can pick r such that $x_1 \notin B_r(x)$. Then, we know that the supremum of x in this ball is less than x_1, meaning that the supremum of f on this ball is less than or equal to $f(x_1)$, meaning that the supremum of f over all possible balls around x is also less than or equal to $f(x_1)$. Similarly, if $x_2 < x$, then the infimum of f over all possible balls around x is greater than or equal to $f(x_2)$. So then, $o(f,x) \leq f(x_1) - f(x_2)$.

Then, suppose we have $x_1 < x_2, ...x_n$. First, assume that $x_1 \neq a, x_n \neq b$. Then, we can find $y_1, y_2, ...y_n$ with $y_1 = a < x_1 < y_2 < x_2 ... < y_n < x_n < b = y_{n+1}$. Then, we know that $o(f, x_i) \leq f(y_{i+1}) - f(y_i)$ for each i. Then:

$$o(f,x_1) + ... + o(f,x_n) \leq f(y_2) - f(a) + f(y_3) - f(y_2) + ... + f(b) - f(y_n) = f(b) - f(a)$$

Since the above sum telescopes.

Now, if $x_1 = a$ or $x_n = b$, note that on all balls, $\inf_{B_r(a)} f = f(a)$, since f is an increasing function, and $\sup_{B_r(a)} f = f(b)$. Then, for $x_1 > a$, we have:

$$o(f,a) \leq f(x_1) - f(a)$$

and for $x_2 < b$, we have:

$$o(f,b) \leq f(b) - f(x_2)$$

In which case we can apply essentially the same telescoping argument as above to find that, indeed, $o(f,x_1) + ...o(f,x_n) \leq f(b) - f(a)$.

Now, we know that the set of discontinuities of f is given by $\bigcup_{n=1}^{\infty}\{x : o(f,x) \geq \frac{1}{n}\}$. I claim that each set $\{x : o(f,x) \geq \frac{1}{n}\}$ is finite. Suppose it wasn't. Then, in particular, there are more than $n(ceil(f(a) - f(b)) + 1)$ unique elements in each set. Denote this number by $c = n(ceil(f(a) - f(b)) + 1)$. Then, take $x_1, ...x_c$. We then have:

$$o(f,x_1) + ... + o(f,x_c) \geq c\frac{1}{n} = (ceil(f(b) - f(a)) + 1) > f(b) - f(a)$$

Which contradicts our earlier conclusion that

$$o(f,x_1) + ... + o(f,x_c) \leq f(b) - f(a)$$

Thus, each set $\{x : o(f,x) \geq \frac{1}{n}\}$ must have a finite number of elements. In fact, each set must have fewer than c elements! So, if each set in the union $\bigcup_{n=1}^{\infty}\{x : o(f,x) \geq \frac{1}{n}\}$ is finite, in particular, each set in the union is measure 0. Then, we know that a countable union of measure 0 sets is also measure 0, so the whole union is measure 0. Since this union is the set of discontinuities of f, we're done.