1) Let A be a rectangle in $\mathbb{R}^n$ and let $f, g : A \to \mathbb{R}$ be integrable.

a) For any partition P of A and any subrectangle S, show that

$$m_S(f) + m_S(g) \leq m_S(f+g) \quad \text{and} \quad M_S(f+g) \leq M_S(f) + M_S(g)$$

and therefore,

$$L(f,P)+L(g,P) \leq L(f+g,P) \quad \text{and} \quad U(f+g,P) \leq U(f,P)+L(g,P)$$

b) Show that $f+g$ is integrable and $\int_A (f+g) = \int_A f + \int_A g$.

c) For any constant c, show that cf is integrable and $\int_A cf = c \in_A f$.

Proof:

a) Note that for any two bounded sets $A \subset B \subset \mathbb{R}$, we have that $\inf(B) \leq \inf A$ as

$$\inf(A) \in A \implies \inf(A) \in B \implies \inf(B) \leq \inf(A)$$

and if $\inf(A) \notin A$, we have that if $A' = A \cup \{\inf(A)\}$, then $\inf(A') = \inf(A)$ so the previous argument holds.

Given any partition P of A and $S \in P$, we have by definition

$$\begin{aligned} m_S(f) + m_S(g) &= \inf\{f(x) + g(y) : x, y \in S\} \\ m_S(f+g) &= \inf\{f(x) + g(x) : x \in S\} \end{aligned}$$

Now if $f(x_0) + g(x_0) \in \{f(x) + g(x) : x \in S\}$, then it is clearly also in $\{f(x) + g(y) : x, y \in S\}$ and so

$$\{f(x) + g(x) : x \in S\} \subset \{f(x) + g(y) : x, y \in S\}$$

which means

$$m_S(f)+m_S(g) = \inf\{f(x)+g(y) : x, y \in S\} \leq \inf\{f(x)+g(x) : x \in S\} = m_S(f+g)$$

Therefore, we can conclude

$$\begin{aligned} L(f,P) + L(g,P) &= \sum_{S \in P} m_S(f) \cdot V(S) + \sum_{S \in P} m_S(g) \cdot V(S) \\ &= \sum_{S \in P} V(S) \cdot \big(m_S(f) + m_S(g)\big) \\ &\leq \sum_{S \in P} V(S) \cdot m_S(f+g) \\ &= L(f+g, P) \end{aligned}$$

A similar argument shows that $A \subset B \subset \mathbb{R}$ implies $\sup(A) \leq \sup_(B)$.

Given any partition P of A and $S \in P$, we also have

$$\begin{aligned} M_S(f) + M_S(G) &= \sup\{f(x) + g(y) : x, y \in S\} \\ M_S(f+g) &= \sup\{f(x) + g(x) : x \in S\} \end{aligned}$$

Similarly, we have

$$\{f(x) + g(x) : x \in S\} \subset \{f(x) + g(y) : x, y \in S\}$$

and so we can conclude that

$$M_S(f+g) \leq M_S(f) + M_S(g)$$

Therefore, we have

$$\begin{aligned} U(f+g, P) &= \sum_{S \in P} M_S(f+g) \cdot V(S) \\ &\leq \sum_{S \in P} \big(M_S(f) + M_S(g)\big) \cdot V(S) \\ &= \sum_{S \in P} M_S(f) \cdot V(S) + \sum_{S \in P} M_S(g) \cdot V(S) \\ &= U(f, P) + U(g, P) \end{aligned}$$

b) Given any $\epsilon > 0$, we can find partitions P_f and P_g of A so that

$$U(f, P_f) - L(f, P_f) < \frac{\epsilon}{2} \quad \text{and} \quad U(g, P_g) - L(g, P_g) < \frac{\epsilon}{2}$$

Let P be a refinement of P_f and P_g. Then by part (a), we have

$$\begin{aligned} & U(f+g, P) - L(f+g, P) \\ \leq\ & \big(U(f, P) - L(f, P)\big) + \big(U(g, P) - L(g, P)\big) \\ <\ & \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ =\ & \epsilon \end{aligned}$$

and so $f + g$ is integrable. This means that for any partition, P, we have

$$L(f+g, P) \leq \int_A (f+g) \leq U(f+g, P)$$

and combining with part (a) gives us

$$L(f,P)+L(g,P) \leq \int_A (f+g) \leq U(f,P)+U(g,P)$$

Since $L(f,P)+L(g,P)$ and $U(f,P)+U(g,P)$ can be made arbitrarily close, it follows that

$$\int_A (f+g) = \int_A f + \int_A g$$

c) Let $c > 0$. Let P be a partition of A so that

$$U(f,P) - L(f,P) < \frac{\epsilon}{c}$$

Then given any $S \in P$, we have

$$\begin{aligned} m_S(cf) &= \inf_{x\in S}\{cf(x)\} &= c\inf_{x\in s}\{f(x)\} &= cm_S(f) \\ M_S(cf) &= \sup_{x\in S}\{cf(x)\} &= c\sup_{x\in S}\{f(x)\} &= cM_S(f) \end{aligned}$$

Then we have

$$\begin{aligned} U(cf,P) - L(cf,P) &= \sum_{S\in P} M_S(cf)\cdot V(S) - \sum_{S\in P} m_S(cf)\cdot V(S) \\ &= \sum_{S\in P} c\big(M_S(f) - m_S(f)\big)\cdot V(S) \\ &= c\sum_{S\in P}\big(M_S(f) - m_S(f)\big)\cdot V(S) \\ &\leq c\frac{\epsilon}{c} \\ &= \epsilon \end{aligned}$$

If $c < 0$, then we have similar inequalities

$$m_S(cf) = cM_S(f) \quad \text{and} \quad M_S(cf) = cm_S(f)$$

Thus choosing a partition P so that

$$U(f,P) - L(f,P) < -\frac{\epsilon}{c}$$

gives us (by a similar calculation) that

$$U(cf,P) - L(cf,P) = -c\big(U(f,P) - L(f,P)\big) < -c\cdot\frac{\epsilon}{-c} = \epsilon$$

If $c = 0$, then cf is a constant function, namely $cf(x) = 0$ and we have proven in class that constant functions are integrable.

Therefore, cf is integrable. If $c = 0$, then $c\int_A f = c\inf_P U(f,P) = \int_A cf$ which is all equal to 0. If $c \neq 0$, then notice that for any partition, P, $cL(f,P)$ and $cU(f,P)$ bound both $c\int_A f$ and $\int_A cf$. Since the upper and lower sums can be made arbitrarily close, we conclude that

$$\int_A cf = c\int_A f$$

2) Let $f : A \to \mathbb{R}$ and let P be a partition of A. Show that f is integrable if and only if for each subrectangle S the function $f|_S$, the restriction of f to S, is integrable, and that in this case, $\int_A f = \sum_{S \in P} \int_S f|_S$.

Proof: ($\Longleftarrow$) Suppose each $f|_S$ is integrable. Then given $\epsilon > 0$, there exist partitions P_S of S so that

$$U(f|_S, P_S) - L(f|_S, P_S) < \frac{\epsilon}{N}$$

where N is the number of subrectangles in P. Extend the edges of each subrectangle of P_S so that it is a partition of A, and let Q be the partition which contains each P_S. In particular, Q refines each P_S so

$$U(f|_S, Q) - L(f|_S, Q) < \frac{\epsilon}{N}$$

Now for each $R \in Q$ with $R \subset S$, we have that $M_R(f) = M_R(f|_S)$ and $m_R(f) = m_R(f|_S)$. Then letting S_i be the subrectangles of P, we have

$$\begin{aligned} U(f,Q) - L(f,Q) &= \sum_{R \in Q} V(R) \cdot \big(M_R(f) - m_R(f)\big) \\ &= \sum_{i=1}^{N} \sum_{\substack{R \in Q \\ R \subset S_i}} V(R) \cdot \big(M_R(f) - m_R(f)\big) \\ &= \sum_{i=1}^{N} \sum_{\substack{R \in Q \\ R \subset S_i}} V(R) \cdot \big(M_R(f|_S) - m_S(f|_S)\big) \\ &< \sum_{i=1}^{N} \frac{\epsilon}{N} \\ &= \epsilon \end{aligned}$$

and so f is integrable.

($\Longrightarrow$) Now suppose f is integrable. Then given any $\epsilon > 0$, there exists some partition Q which refines P and satisfies

$$U(f,Q) - L(f,Q) < \epsilon$$

Let $S \in P$ and let $P_S \subset Q$ be the set of subrectangles of Q which partitions S. Note that $f = f|_S$ on S so their upper and lower sums are all equal as well. Then similar to the converse, we have

$$U(f,Q) - L(f,Q) = \sum_{i=1}^{N} \sum_{R \in P_S} V(R) \cdot \big(M_R(f|_S) - m_S(f|_S)\big) < \epsilon$$

Since $V(R)$ and $M_R(f|_S) - m_S(f|_S)$ are both positive, their product is also positive which means that

$$\sum_{R \in P_S} V(R) \cdot \big(M_R(f|_S) - m_S(f|_S)\big) \leq \sum_{i=1}^{N} \sum_{R \in P_S} V(R) \cdot \big(M_R(f|_S) - m_S(f|_S)\big) < \epsilon$$

which means $f|_S$ is integrable.

Note that if $f|_S$ is integrable, then the extension $g_S : A \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

is also integrable since given any partition of S, we can extend it to a partition of A, and the subrectangles that do not cover S have equal upper and lower sums, namely 0, and so their difference do not contribute anything to the upper and lower sums of subrectangles covering S. In particular, this means $\int_S f|_S = \int_A g_S$. Clearly, we have $f = \sum_{S \in P} g_S$ and by problem 3-3 of Spivak, we have

$$\int_A f = \sum_{S \in P} \int_A g_S = \sum_{S \in P} \int_S f|_S$$

3) Let $f, g : A \to \mathbb{R}$ be integrable and suppose $f \leq g$. Show that $\int_A f \leq \int_A g$.

Proof: First we show that if an integrable function $h : A \to \mathbb{R}$ satisfies $h(x) \geq 0$ for all $x \in A$, then $\displaystyle\int_A h \geq 0$. Let P be any partition of A and let $S \in P$. Then note that $0 \leq h(x)$, for all $x \in S \subset A$ so $m_S(h) \geq 0$. This means that $L(h, P) \geq 0$ and since h is integrable, we have

$$0 \leq L(h, P) \leq \sup_P L(h, P) = \int_A h$$

Now let $h : A \to \mathbb{R}$ be the function $h(x) = g(x) - f(x)$. By problem 1 of this problem set, h is integrable and

$$\int_A h = \int_A g - \int_A f$$

Since $g(x) \geq f(x)$ for all $x \in A$, it follows that $h(x) \geq 0$ and so

$$0 \leq \int_A g - \int_A f \implies \int_A f \leq \int_A g$$

4) If $f : A \to \mathbb{R}$ is integrable, show that $|f|$ is integrable and $|\int_A f| \le \int_A |f|$.

Proof: f is integrable so given any $\epsilon > 0$, there exists some partition P of A so that

$$U(f,P) - L(f,P) < \epsilon$$

Given any $S \in P$, we have three possibilities:

1) $m_S(f) \le M_S(f) \le 0$
2) $m_S(f) \le 0 \le M_S(f)$
3) $0 \le m_S(f) \le M_S(f)$

In case (1), we have $m_S\big(|f|\big) = \big|M_S(f)\big|$ and $M_S\big(|f|\big) = \big|m_S(f)\big|$, and so

$$M_S\big(|f|\big) - m_S\big(|f|\big) = M_S(f) - m_S(f)$$

In case (3), we have $m_S\big(|f|\big) = m_S(f)$ and $M_S\big(|f|\big) = M_S(f)$ and so

$$M_S\big(|f|\big) - m_S\big(|f|\big) = M_S(f) - m_S(f)$$

In case (2), we have $M_S\big(|f|\big) = M_S(f)$ and $0 \le m_S\big(|f|\big) \le \big|m_S(f)\big|$ and combining and rearranging gives us

$$m_S(f) \le -m_S\big(|f|\big) \le 0 \le M_S\big(|f|\big) = M_S(f)$$

and so we get the inequality

$$M_S\big(|f|\big) - m_S\big(|f|\big) \le M_S(f) - m_S(f)$$

Thus we have

$$\begin{aligned} U\big(|f|,P\big) - L\big(|f|,P\big) &= \sum_{S\in P} V(S)\cdot\big(M_S\big(|f|\big) - m_S\big(|f|\big)\big) \\ &\le \sum_{S\in P} V(S)\cdot\big(M_S(f) - m_S(f)\big) \\ &< \epsilon \end{aligned}$$

and so $|f|$ is integrable.

To show that $\big|\int_A f\big| \le \int_A |f|$, note that $-|f| \le f \le |f|$ and by problem 3-5 of Spivak, we have

$$-\int_A |f| \le \int_A f \le \int_A |f| \implies \left|\int_A f\right| \le \int_A |f|$$

5a) Show that an unbounded set cannot have content 0.

b) Give an example of a closed set of measure 0 which does not have content 0.

<u>Proof</u>:

a) Suppose $A \in \mathbb{R}^n$ has content zero. Then given any $\epsilon > 0$, there exist finitely many open rectangles $R_1, \ldots, R_k$ which cover A. Each R_i is a rectangle and so is bounded by the ball $B(0, r_i)$ for some r_i. Let $r = \max\{r_1, \ldots, r_i\}$. Then

$$A \subset \bigcup_{i=1}^{k} R_i \subset B(0, r)$$

and so A is bounded.

b) Consider the subset of $\mathbb{R}^2$

$$A = \{(x, 0) : x \in \mathbb{N}\} = \{(0,0), (1,0), (2,0), \ldots\}$$

Define sets

$$\begin{aligned} B_i &= (i, i+1) \times \mathbb{R} \quad \text{for } i \in \mathbb{N} \\ C_- &= \{(x,y) \in \mathbb{R}^2 : y < 0\} \\ C_+ &= \{(x,y) \in \mathbb{R}^2 : y > 0\} \\ D &= \{(x,y) \in \mathbb{R}^2 : x < 0\} \end{aligned}$$

Then each of these sets are open and their union

$$X = \left(\bigcup_{i=0}^{\infty} B_i \right) \cup C_- \cup C_+ \cup D$$

is also open. Since $A = \mathbb{R}^2 - X$, we have that A is indeed closed. Since A is also countable, we have that it is measure zero. Since it is unbounded, it does not have content zero.

6) Let $A = \bigcup_{i=1}^{\infty}(a_i, b_i)$ be a countable union of open intervals, and assume that $\big([0,1] \cap \mathbb{Q}\big) \subset A$. Show that if $\sum_{i=1}^{\infty}(b_i - a_i) < 1$ then the boundary of A is not of measure 0.

Proof: Without loss of generality, we may assume that $A \subset [0,1]$. This is because if A' is some set which contains A, restricting to $A' \cap [0,1]$ and taking the boundary gives us

$$\text{bd}(A) = \text{bd}\big(A' \cap [0,1]\big) \subset \text{bd}(A') \cup \{0,1\}$$

and so if $\text{bd}(A)$ does not have measure 0, neither will $\text{bd}(A')$.

Additionally, we may assume that $\{0,1\} \notin A$ as removing two points would only change the boundary by finitely many points, which does not affect whether the set is measure zero or not. More precisely, if $A' = A - \{0,1\}$, then problem 1-18 of Spivak

$$\text{bd}(A') = [0,1] - A' = [0,1] - \big(A - \{0,1\}\big) = \big([0,1] - A\big) \cup \{0,1\}$$

So by problem 1-18 of Spivak, we have that $\text{bd}(A) = [0,1] - A$. Since $A \subset [0,1]$, it follows that

$$\text{bd}(A) \cup A = \big([0,1] - A\big) \cup A = [0,1]$$

Suppose $S := \displaystyle\sum_{i=1}^{\infty}(b_i - a_i) = 1 - \epsilon$, and suppose $\text{bd}(A)$ has measure zero. Then since $A = \displaystyle\bigcup_{i=1}^{\infty}(a_i, b_i)$, it follows that A is covered by all the (a_i, b_i), and so A has a "length" of at most $1 - \epsilon$. Then since $\text{bd}(A)$ is measure zero, it can be covered with open intervals whose sum is less than $\dfrac{\epsilon}{2}$, which means the "length" of $\text{bd}(A)$ is less than $\dfrac{\epsilon}{2}$. Since the length of $[0,1]$ is 1, it follows that

$$1 \leq S + \frac{\epsilon}{2} = (1 - \epsilon) + \frac{\epsilon}{2} = 1 - \frac{\epsilon}{2} < 1$$

which is a contradiction.

7) Let $f : [a, b] \to \mathbb{R}$ be an increasing function. Show that the set of discontinuities of f is of measure 0.

Spivak Problem 1-30: Given any increasing $f : [a, b] \to \mathbb{R}$ and given distinct $x_1, \ldots, x_n \in [a, b]$, we have

$$\sum_{i=1}^{n} o(f, x_i) < f(b) - f(a)$$

Proof: By the definition of the oscillation, we have

$$o(f, x_i) = \lim_{r\to 0^+} o\big(f, (x_i - r, x_i + r)\big) = \lim_{r\to 0^+}\left(\sup_{x\in(x_i-r,x_i+r)} f(x) - \inf_{x\in(x_i-r,x_i+r)} f(x)\right)$$

Since f is increasing, we have

$$o(f, x_i) = \lim_{r\to 0^+}\big(f(x_i + r) - f(x_i - r)\big)$$

Assume $x_1 < \cdots < x_n$ and choose $r > 0$ small enough so that none of any $f(x_i + r)$ or $f(x_j - r)$ are equal, for any i and j. Then there exist midpoints between each consecutive pair of $f(x_i)$'s, say $x_i < x_{m_i} < x_{i+1}$ for $1 \le i \le n - 1$. Since f is increasing, we have for $2 \le i \le n - 1$, that $f(x_i + r) < f(x_{m_i})$ and $f(x_i - r) > f(x_{m_{i-1}})$, and that $f(x_n - r) > f(x_{m_{n-1}})$ and $f(x_1 + r) < f(x_{m_1})$, so we get a telescoping sum

$$\begin{aligned}
&\sum_{i=1}^{n}\big(f(x_i + r) - f(x_i - r)\big)\\
<\ & f(x_n + r) + \left(\sum_{i=2}^{n-1}\big(-f(x_{m_i}) + f(x_{m_1})\big)\right) - f(x_1 - r)\\
=\ & f(x_n + r) - f(x_1 - r)\\
\le\ & f(b) - f(a)
\end{aligned}$$

This proves problem 1-30.

Problem Proof: Now given $n \in \mathbb{N}$, consider the set

$$D_n = \left\{x \in [a, b] : o(f, x) > \frac{1}{n}\right\}$$

Then the set, D, of discontinuities of f is the union of all D_n. Suppose D_n is infinite. Pick distinct $x_1, \ldots, x_m \in D_n$ where

$$n\big(f(b) - f(a)\big) < m \implies \frac{m}{n} > f(b) - f(a)$$

Then by Spivak's Problem 1-30, we have

$$\sum_{i=1}^{m} o(f, x_i) > \sum_{i=1}^{m} \frac{1}{n} = \frac{m}{n} > f(b) - f(a)$$

which contradicts the same problem in Spivak. Thus D_n is necessarily finite. Since D is a countable union of finite sets, D itself is countable and so is of measure zero.