1) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a C^1 function; we write f in the form $f(x, y_1, y_2)$. Assume that $f(3, -1, 2) = 0$ and

$$f'(3, -1, 2) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

a) Show that there is a function $g : B \to \mathbb{R}^2$ defined on an open set B in $\mathbb{R}$ such that $e \in B$ and such that $g(3) = (-1, 2)$ and

$$f(x, g_1(x), g_2(x)) = 0$$

for $x \in B$.

b) Find $g'(3)$.

c) Discuss the problem of solving the equation $f(x, y_1, y_2) = 0$ for an arbitrary pair of the unknowns in terms of the third, near the point $(3, -1, 2)$.

Solution:

a) f is C^1 so it is C^1 in some neighbourhood around $(3, -1, 2)$. If we denote $\dfrac{\partial f}{\partial y}$ to be the matrix

$$\frac{\partial f}{\partial y} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

then we have that $\dfrac{\partial f}{\partial y}$ is invertible near $(3, -1, 2)$ since $\det \dfrac{\partial f}{\partial y} = 3 \neq 0$. Since $f(3, -1, 2) = 0$, we have by the implicit function theorem that there exists an open $B \subset \mathbb{R}$ with $3 \in B$, and there exists an open $A \subset \mathbb{R}^2$ with $(-1, 2) \in A$ such that there is some unique function $\overline{g} : B \to A$ with the properties

$$\overline{g}(3) = (-1, 2) \quad \text{and} \quad \forall x \in B,\ f\big(x, \overline{g}(x)\big) = f\big(x, \overline{g_1}(x), \overline{g_2}(x)\big) = 0$$

Let $g : B \to \mathbb{R}^2$ be the function $g = \overline{g}$. Then this function solves the problem.

b) Let $H : \mathbb{R}^3 \to \mathbb{R}^3$ be the function

$$H(x, y_1, y_2) = \big(x, f(x, y_1, y_2)\big)$$

and let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the function

$$\pi(x, y_1, y_2) = (y_1, y_2)$$

Then g from part (a) is

$$g(x) = \pi \circ H^{-1}(x, 0, 0)$$

by the implicit function theorem. Note that by the inverse function theorem, we have

$$\begin{aligned} DH^{-1}(3,0,0) &= \left[H'\big(H^{-1}(3,0,0)\big)\right]^{-1} \\ &= \left[H'(3,-1,2)\right]^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -1/3 \\ -1 & 1/3 & 2/3 \end{pmatrix} \end{aligned}$$

Thus by the chain rule, we have

$$g'(3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -1/3 \\ -1 & 1/3 & 2/3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

EDIT: The final result is actually supposed to be $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ rather than $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Everything else is fine.

c) The other potential candidates would be to solve for (x, y_1) in terms of y_2 or to solve for (x, y_2) in terms of y_1. The first case is very similar to part (a) since

$$\frac{\partial f}{\partial(x, y_1)} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

is the same as $\dfrac{\partial f}{\partial y}$ but with two columns swapped. Since swapping the columns only changes the determinant possibly by a sign, it is still invertible and so we can still apply the implicit function theorem. In the second case however, we have

$$\frac{\partial f}{\partial(x, y_2)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

which is not invertible and so we cannot apply the implicit function theorem.

2) Let $f : R^2 \to \mathbb{R}$ be C^1, with $f(2,-1) = -1$. Set

$$G(x,y,u) = f(x,y) + u^2,$$
$$H(x,y,u) = ux + 3y^3 + u^3.$$

The equations $G(x,y,u) = 0$ and $H(x,y,u) = 0$ have the solution $(x,y,u) = (2,-1,1)$.

a) **What conditions on f' ensure that there are C^1 functions $x = g(y)$ and $u = h(y)$ defined on an open set in $\mathbb{R}$ that satisfy both equations, and such that $g(-1) = 2$ and $h(-1) = 1$?**

b) **Under the conditions of (a) and assuming that $f'(2,-1) = (1 \quad -3)$, find $g'(-1)$ and $h'(-1)$.**

<u>Solution</u>:

a) Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be the function

$$F(x,y,u) = \begin{pmatrix} G(x,y,u) \\ H(x,y,u) \end{pmatrix} = \begin{pmatrix} f(x,y) + u^2 \\ ux + 3y^3 + u^3 \end{pmatrix}$$

Then $F(2,-1,1) = 0$. We can find the differential of F to get

$$\begin{aligned} F'(x,y,u) &= \begin{pmatrix} \partial_1 G(x,y,u) & \partial_2 G(x,y,u) & \partial_3 G(2,y,u) \\ \partial_1 H(x,y,u) & \partial_2 H(x,y,u) & \partial_3 H(x,y,u) \end{pmatrix} \\ &= \begin{pmatrix} \partial_1 f(x,y) & \partial_2 f(x,y) & 2u \\ u & 9y^2 & x + 3u^2 \end{pmatrix} \end{aligned}$$

and so $F'(2,-1,1) = \begin{pmatrix} \partial_1 f(2,-1) & \partial_2 f(2,-1) & 2 \\ 1 & 9 & 5 \end{pmatrix}$. If we want to apply the implicit function theorem near $(2,-1,1)$ to solve (x,u) in terms of y, we need the matrix

$$M = \frac{\partial F}{\partial(x,u)} = \begin{pmatrix} \partial_1 f(2,-1) & 2 \\ 1 & 5 \end{pmatrix}$$

to be invertible, that is, $\partial_1 f(2,-1) \neq \frac{2}{5}$. If this were the case, then applying the implicit function theorem would give us some $\phi : A \to B$ for open $-1 \in A \subset \mathbb{R}$ and $(2,1) \in B \subset \mathbb{R}^2$ such that $\phi(-1) = (2,1)$. Let $g = \phi_1$ and $h = \phi_2$.

b) Let $\Gamma : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$\Gamma(x, y, u) = \big(G(x, y, u), y, H(x, y, u)\big)$$

and $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$\pi(x, y, u) = (x, u)$$

By the implicit function theorem, we have that ϕ from (a) is given by

$$\phi(y) = \pi\big(\Gamma^{-1}(0, y, 0)\big)$$

By the inverse function theorem, we have that

$$\begin{aligned}
(\Gamma^{-1})'(0, -1, 0) &= \big[\Gamma'\big(\Gamma^{-1}(0, -1, 0)\big)\big]^{-1} \\
&= \big[\Gamma'(2, -1, 1)\big]^{-1} \\
&= \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 0 \\ 1 & 9 & 5 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \frac{5}{3} & 11 & -\frac{2}{3} \\ 0 & 1 & 0 \\ -\frac{1}{3} & -4 & \frac{1}{3} \end{pmatrix}
\end{aligned}$$

So by the chain rule, we have

$$\begin{aligned}
\phi'(-1) = \begin{pmatrix} g'(-1) \\ h'(-1) \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot (\Gamma^{-1})'(0, -1, 0) \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 11 \\ -4 \end{pmatrix}
\end{aligned}$$

Thus $g'(-1) = 11$ and $h'(-1) = -4$.

3) Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be C^1 functions. "In general", one expects that each of the equations $f(x,y,z) = 0$ and $g(x,y,z) = 0$ represents a "nice" surface in $\mathbb{R}^3$ and that their intersections is a smooth curve. Show that if (x_0, y_0, z_0) satisfies both of these equations, and if $\partial(f,g)/\partial(x,y,z)$ has rank 2 at $p_0 = (x_0, y_0, z_0)$, then near p_0 one can solve these equations for two of x, y, z in terms of the third, thus representing the solution set locally as a parametrized curve.
Note: There is only one reasonable way to interpret the notation $\partial(f,g)/\partial(x,y,z)$.

Solution: Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be the function defined by

$$F(x_1, x_2, x_3) = \big(f(x_1, x_2, x_3), g(x_1, x_2, x_3)\big)$$

Then $\dfrac{\partial(f,g)}{\partial(x_1, x_2, x_3)}$ is the differential of F, that is

$$\frac{\partial(f,g)}{\partial(x_1, x_2, x_3)} = F'(x_1, x_2, x_3) = \begin{pmatrix} \partial_1 f(x_1, x_2, x_3) & \partial_2 f(x_1, x_2, x_3) & \partial_3 f(x_1, x_2, x_3) \\ \partial_1 g(x_1, x_2, x_3) & \partial_2 g(x_1, x_2, x_3) & \partial_3 g(x_1, x_2, x_3) \end{pmatrix}$$

To simplify notation, let $\partial_i f(x_1, x_2, x_3) = \phi_i$ and $\partial_i g(x_1, x_2, x_3) = \gamma_i$. Let $p_0 = (x_1, x_2, x_3)$ be so that $F(p_0) = 0$ and that $F'(p_0)$ has rank 2. This means that we can row reduce $F'(x_1, x_2, x_3)$ and swap the columns to get a matrix of the form

$$\begin{pmatrix} 1 & 0 & \phi \\ 0 & 1 & \gamma \end{pmatrix}$$

for some $\phi, \gamma \in \mathbb{R}$. Note that we can remove ϕ and γ from the matrix to have the identity which is invertible, and so we could remove the corresponding column from $F'(x_1, x_2, x_3)$ to get an invertible matrix

$$\begin{pmatrix} \phi_j & \phi_k \\ \gamma_j & \gamma_k \end{pmatrix}$$

for some $j, k \in \{1, 2, 3\}$ since row operations preserves invertibility. From here, we can apply the implicit function theorem to get a function $G : A \to \mathbb{R}^2$ for some open $x_i \in A \subset \mathbb{R}$ where $i \neq j, k$, such that

$$G(x_i) = (x_j, x_k)$$

Thus we can solve f and g near p_0 for x_j and x_k in terms of x_i.

4) Let $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ be a C^1 funciton; suppose that $f(a) = 0$ and that $f'(a)$ has rank n. Show that if c is a point of $\mathbb{R}^n$ sufficiently close to 0, then the equation $f(x) = c$ has a solution.

Solution: Since $f'(a)$ has rank n, this means that we can rearrange the columns of $f'(a)$ to get something of the form

$$f'(a) = \begin{pmatrix} \partial_x f(a) & \partial_y f(a) \end{pmatrix}$$

such that $\partial_y f(a)$ is invertible and has rank n. Since column permutations does not change the determinant, except possibly by a sign, we can assume without loss of generality that f is of the form $f(x, y)$, where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$. Thus let $a = (a_x, a_y)$ and so we have $f(a_x, a_y) = 0$ and $\partial_y f(a_x, a_y)$ is invertible.

Define a function $g : \mathbb{R}^{n+n} \to \mathbb{R}^n$ by

$$g(c, y) = f(a_x, y) - c$$

Note that $g(0, a_y) = f(a_x, a_y) - 0 = 0$ and

$$\frac{\partial g}{\partial y}(0, a_y) = \partial_y f(a_x, a_y)$$

which is invertible. Thus by the implicit function theorem, there exist open $A, B \subset \mathbb{R}^n$ with $0 \in A$ and $a_y \in B$, and a unique function $h : A \to B$ such that $h(0) = a_y$ and for all $c \in A$, we have $g\big(c, h(c)\big) = 0$. Expanding this out gives us

$$0 = g\big(c, h(c)\big) = f\big(a_x, h(c)\big) - c \implies f\big(a_x, h(c)\big) = c$$

So for c sufficiently close to 0, that is, for $c \in A$, we have that the point $(x, y) = \big(a_x, h(c)\big)$ satisfies $f(x, y) = c$.