1) Let $A \subset \mathbb{R}^n$ be an open set and $f : A \to \mathbb{R}^n$ a continuously differentiable $1-1$ function such that $f'(x)$ is invertible for all $x \in A$. Show that $f(A)$ is an open set and that $f^{-1} : f(A) \to A$ is differentiable. Show also that $f(B)$ is open for any open set $B \subset A$.

Proof: First we show that $f(A)$ is open. Let $y \in f(A)$. Then by definition there exists some $x \in A$ such that $y = f(x)$. Since $f'(x)$ is invertible, the inverse function theorem says that there exist open sets $V, W \in \mathbb{R}^n$ where $x \in V \subset A$ and $y = f(x) \in W$ such that $f : V \to W$ is invertible. Since $W = f(V)$ and $V \subset A$, we have $y \in W \subset f(A)$ and so $f(A)$ is open.

Now we show that $f^{-1} : f(A) \to A$ is differentiable. First note that f^{-1} exists and is well-defined since f is injective and we have restricted the codomain to the range of f. Now given any $x \in A$, we have that $f'(x)$ is invertible and so the inverse function theorem says that there exists open sets $V_x, W_y \in \mathbb{R}^n$ where $x \in V_x \subset A$ and $y = f(x) \in W_y \subset f(A)$ such that $f_x : V \to W$ has a continuously differentiable inverse $f_x^{-1} : W_y \to V_x$. Note that each f_x^{-1} must agree with f^{-1} since f is injective. In particular, $f_{x_1}^{-1}$ must agree with $f_{x_2}^{-1}$ on $W_{y_1} \cap W_{y_2}$ because f is injective. Since the functions agree, we have that

$$(f_{x_1}^{-1})'(y) = (f_{x_2}^{-1})'(y)$$

for any $x_1, x_2 \in A$ and any $y \in W_{x_1} \cap W_{x_2}$. Since $\displaystyle\bigcup_{x \in A} V_x = A$ and $\displaystyle\bigcup_{y \in f(A)} W_y = f(A)$, we have that $(f^{-1})'(y)$ given by

$$(f^{-1})'(y) := (f_{x_0}^{-1})'(y)$$

for some $x_0 \in A$ is well-defined which means f^{-1} is differentiable.

Now we will show that

$$B \subset A \text{ open } \implies f(B) \subset f(A) \text{ open}$$

Consider the restriction of f to some open B

$$f_B : B \to f(B), \quad f_B(x) = f(x)$$

Then f_B is continuously differentiable and injective since f is, and $f_B'(x)$ is invertible since $f'(x)$ is. Thus applying the first part of this problem says that $f(B)$ is open.

2a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Show that f is *not* $1-1$.

b) Generalize this result to the case of a continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}^m$, where $n > m$.

Proof:

a) Suppose for the sake of a contradiction that f is injective. Then one of $D_1 f(x, y) \neq 0$ or $D_2 f(x, y) \neq 0$ is true as otherwise f would be constant which is not injective. Without loss of generality, suppose $D_1 f(a, b) \neq 0$ for some (a, b) in some open set A. Define a function $g : A \to \mathbb{R}^2$ by

$$g(x, y) = \big(f(x, y), \pi(x, y)\big) = \big(f(x, y), y\big)$$

Since g is a composition of continuously differentiable functions, g itself is continuously differentiable with differential

$$g'(x, y) = \begin{pmatrix} D_1 f(x, y) & D_2 f(x, y) \\ 0 & 1 \end{pmatrix}$$

and so $\det g'(x, y) = D_1 f(x, y) \neq 0$, for all $(x, y) \in A$. By Spivak problem 2-36, we have that $g(A)$ is open. Thus we can find some $r \in \mathbb{R}$ such that $g(a, b) \in B\big((a, b), r\big) \subset g(A)$. Note additionally that for any $(u, v) \in g(A)$, we have that there exists some $(x, y) \in A$ such that

$$(u, v) = g(x, y) = \big(f(x, y), y\big)$$

However, the point $\left(f(a, b), b + \dfrac{r}{2}\right)$ is in $B\big((a, b), r\big)$ but is not contained in $g(A)$, which is a contradiction.

b) Let $n = m + k$. Denote a point of $\mathbb{R}^n$ by

$$z = (x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_k)$$

Now given $f : \mathbb{R}^{m+k} \to \mathbb{R}^m$ continuously differentiable where $f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$, define four matrices

$$\begin{aligned}
D_{1,m}f(z) &:= \begin{pmatrix} D_1f_1(z) & \dots & D_mf_1(z) \\ \vdots & \ddots & \vdots \\ D_1f_m(z) & \dots & D_mf_m(z) \end{pmatrix} \\
D_{m,n}f(z) &:= \begin{pmatrix} D_{m+1}f_1(z) & \dots & D_{m+k}f_1(z) \\ \vdots & \ddots & \vdots \\ D_{m+1}f_m(z) & \dots & D_{m+k}f_m(z) \end{pmatrix} \\
\overline{0} &:= \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \in M_{k,m}(\mathbb{R}) \\
I &:= \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \in M_{k,k}(\mathbb{R})
\end{aligned}$$

For the sake of a contradiction, suppose f is injective. Then $Df(z) \neq 0$ as otherwise, f is constant and not injective. Then for some $D_if_j(z)$, we have that it is nonzero. Without loss of generality, suppose

$$D_1f_1(z_0) \neq 0$$

in some open set $A \in \mathbb{R}^{m+k}$. Now define another function $\pi : \mathbb{R}^{m+k} \to \mathbb{R}^k$ by

$$\pi(x, y) = y = (y_1, \dots, y_k)$$

and define $g : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(x, y) = \big(f_1(x, y), \dots, f_m(x, y), y_1, \dots, y_k\big) = \big(f(x, y), \pi(x, y)\big)$$

Then g is a composition of continuously differentiable functions and so is itself continuously differentiable. Thus by the chain rule we get

$$Dg(z) = \begin{pmatrix} D_{1,m}f(z) & D_{m,n}f(z) \\ \overline{0} & I \end{pmatrix}$$

and so $\det Dg(z) = \det\big(D_{1,m}f(z)\big) \det\big(I\big) = \det D_{1,m}f(z)$. This value is necessarily nonzero as otherwise f would not be injective. Thus by Spivak problem 2-36, $g(A)$ is open. Similarly to the part (a) of this question, every point of $g(A)$ is of the form

$$g(x, y) = \big(f(x, y), y\big)$$

and there exists an open ball B of radius ϵ such that $g(x, y) \in B \subset g(A)$. Then taking the point $z' = g\left(x, y + \frac{1}{\epsilon} e_n\right)$ gives us $z' \in B$ but $z' \notin g(A)$ which is a contradiction.

3a) If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f'(a) \neq 0$ for all $a \in \mathbb{R}$, show that f is $1-1$ on $\mathbb{R}$.

b) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (e^x \cos y, e^x \sin y)$. Show that $f'(x,y)$ is always invertible yet f is not $1-1$.

Proof:

a) We will prove the contrapositive. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function that is not injective. Then there exists $x, y \in \mathbb{R}$ such that

$$x \neq y \quad \text{and} \quad f(x) = f(y)$$

Without loss of generality, assume $x < y$. Since f is continuous and differentiable on $\mathbb{R}$, it is in particular continuous on $[x, y]$ and differentiable on (x, y), and so there exists some $a \in (x, y)$ such that

$$f(y) - f(x) = f'(a) \cdot (y - x)$$

The left hand side equals to 0 but $y - x \neq 0$ and so we conclude that $f'(a)$ must necessarily be 0.

b) Note that calculating the partials of f gives us the differential

$$f'(x,y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

Then the determinant of $f'(x,y)$ is given by

$$\det f'(x,y) = (e^x \cos y)^2 + (e^x \sin y)^2 = (e^x)^2(\cos^2 y + \sin^2 y) = e^{2x} > 0$$

In particular, we have that $\det f'(x,y)$ is never zero and so is always invertible. However, consider the points $(0,0)$ and $(0, 2\pi)$. Then

$$f(0,0) = f(0, 2\pi) = (1, 0)$$

but clearly

$$(0,0) \neq (0, 2\pi)$$

Thus f is not $1-1$.