MAT257 Assignment 3 Solution

1. *Proof.* Suppose f is differentiable at $a \in \mathbb{R}^n$. Then there exists a linear map $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that

$\lim\limits_{h \to 0} \dfrac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$. By the limit law about products of limits, we have:

$$\lim_{h \to 0} |f(a+h) - f(a) - \lambda(h)| = \lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} \cdot |h| = \lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} \cdot \lim_{h \to 0} |h| = 0 \cdot 0 = 0$$

Consider $|f(a+h) - f(a)|$. By the triangle inequality, we have:

$$|f(a+h) - f(a)| = |f(a+h) - f(a) - \lambda(h) + \lambda(h)| \leq |f(a+h) - f(a) - \lambda(h)| + |\lambda(h)|$$

Since λ is linear, λ is continuous. So in particular, $\lim\limits_{h \to 0} \lambda(h) = \lambda(0) = 0$.

Because $\lim\limits_{h \to 0} |f(a+h) - f(a) - \lambda(h)| = 0$, it follows that

$$\lim_{h \to 0} |f(a+h) - f(a) - \lambda(h)| + |\lambda(h)| = \lim_{h \to 0} |f(a+h) - f(a) - \lambda(h)| + \lim_{h \to 0} |\lambda(h)| = 0 + 0 = 0.$$

Since $0 \leq |f(a+h) - f(a)| \leq |f(a+h) - f(a) - \lambda(h)| + |\lambda(h)|$, by the squeeze theorem $\lim\limits_{h \to 0} |f(a+h) - f(a)|$

must be 0, which implies f is continuous at a.

□

2. *Proof of (a).* If $x = 0$, then $h(t) = f(t \cdot 0) = f(0) = 0$ is constant so it is certainly differentiable.

Suppose $x \neq 0$. We first show h is differentiable at 0. $h(0) = f(0 \cdot x) = f(0) = 0$. Thus

$$\lim_{t \to 0^+} \frac{h(t) - h(0)}{t} = \lim_{t \to 0^+} \frac{f(tx) - 0}{t} = \lim_{t \to 0^+} \frac{|tx| \cdot g(\frac{tx}{|t||x|})}{t} = \lim_{t \to 0^+} \frac{t|x| \cdot g(\frac{tx}{t|x|})}{t} = |x| g(\frac{x}{|x|})$$

Similarly, since $g(-x) = -g(x)$, it follows that

$$\lim_{t \to 0^-} \frac{h(t) - h(0)}{t} = \lim_{t \to 0^-} \frac{f(tx) - 0}{t} = \lim_{t \to 0^-} \frac{|tx| \cdot g(\frac{tx}{|t||x|})}{t} = \lim_{t \to 0^-} \frac{-t|x| \cdot g(\frac{tx}{-t|x|})}{t} = -|x| \cdot g(\frac{x}{-|x|}) = |x| g(\frac{x}{|x|})$$

So h is differentiable at 0 with $h'(0) = |x| g(\frac{x}{|x|})$.

Now let $t \in \mathbb{R}$ be nonzero. Suppose $t > 0$. Then we can choose $s \in \mathbb{R}$ to be such that $|s|$ is sufficiently small so that $t + s > 0$. Hence

$$\begin{aligned}
\lim_{s \to 0} \frac{h(t+s) - h(t)}{t} &= \lim_{s \to 0} \frac{f((t+s)x) - f(tx)}{s} = \lim_{s \to 0} \frac{|t+s||x| \cdot g(\frac{(t+s)x}{|t+s||x|}) - |t||x| \cdot g(\frac{tx}{|t||x|})}{s} \\
&= \lim_{s \to 0} \frac{(t+s)|x| \cdot g(\frac{(t+s)x}{(t+s)|x|}) - t|x| \cdot g(\frac{tx}{t|x|})}{s} = \lim_{s \to 0} \frac{(t+s)|x| \cdot g(\frac{x}{|x|}) - t|x| \cdot g(\frac{x}{|x|})}{s} \\
&= \lim_{s \to 0} \frac{s|x| \cdot g(\frac{x}{|x|})}{s} = |x| g(\frac{x}{|x|})
\end{aligned}$$

So h is differentiable at t when $t > 0$ with $h'(t) = |x| g(\frac{x}{|x|})$.

Suppose $t < 0$. Similarly as above, we can choose $s \in \mathbb{R}$ to be such that $|s|$ is sufficiently small so that $t + s < 0$. Apply $g(-x) = -g(x)$ and we have:

$$\begin{aligned}
\lim_{s \to 0} \frac{h(t+s) - h(t)}{t} &= \lim_{s \to 0} \frac{f((t+s)x) - f(tx)}{s} = \lim_{s \to 0} \frac{|t+s||x| \cdot g(\frac{(t+s)x}{|t+s||x|}) - |t||x| \cdot g(\frac{tx}{|t||x|})}{s} \\
&= \lim_{s \to 0} \frac{-(t+s)|x| \cdot g(\frac{(t+s)x}{-(t+s)|x|}) + t|x| \cdot g(\frac{tx}{-t|x|})}{s} = \lim_{s \to 0} \frac{(t+s)|x| \cdot g(\frac{x}{|x|}) - t|x| \cdot g(\frac{x}{|x|})}{s} \\
&= \lim_{s \to 0} \frac{s|x| \cdot g(\frac{x}{|x|})}{s} = |x| g(\frac{x}{|x|})
\end{aligned}$$

Thus h is differentiable at t when $t < 0$ with $h'(t) = |x| g(\frac{x}{|x|})$ and this proves h is differentiable on $\mathbb{R}$. □

Proof of (b). If $g = 0$, then $f = 0$ so clearly f is differentiable at 0.

Suppose $g \neq 0$. Assume f is differentiable at $(0,0)$ and $\lambda : \mathbb{R}^2 \to \mathbb{R}$ is its differential at $(0,0)$. Then

$$\lim_{(h,k)\to 0} \frac{|f(h,k) - f(0,0) - \lambda(h,k)|}{|(h,k)|} = \lim_{(h,k)\to 0} \frac{\left| |(h,k)| g(\frac{(h,k)}{|(h,k|)}) - \lambda(h,k) \right|}{|(h,k)|} = 0$$

. In particular, since $g(1,0) = 0$ and $g(-1,0) = -g(1,0) = 0$, we have:

$$\lim_{(h,0)\to 0} \frac{\left| |h| g(\frac{(h,0)}{|h|}) - \lambda(h,0) \right|}{|h|} = \lim_{(h,0)\to 0} \frac{\left| |h| g(\pm 1, 0) - \lambda(h,0) \right|}{|h|} = \lim_{(h,0)\to 0} \frac{|\lambda(h,0)|}{|h|} = 0 \quad \bigstar$$

. Similarly, since $g(0,1) = 0$ and $g(0,-1) = -g(0,1) = 0$, we have:

$$\lim_{(0,k)\to 0} \frac{\left| |k| g(\frac{(0,k)}{|k|}) - \lambda(0,k) \right|}{|k|} = \lim_{(0,k)\to 0} \frac{\left| |k| g(0, \pm 1) - \lambda(0,k) \right|}{|k|} = \lim_{(0,k)\to 0} \frac{|\lambda(0,k)|}{|k|} = 0 \quad \bigstar\bigstar$$

By linearity of λ, it follows that

$$\frac{|\lambda(h,k)|}{|(h,k)|} = \frac{|\lambda(h,0) + \lambda(0,k)|}{|(h,k)|} \leq \frac{|\lambda(h,0)| + |\lambda(0,k)|}{|(h,k)|} \leq \frac{|\lambda(h,0)|}{|h|} + \frac{|\lambda(0,k)|}{|k|}$$

By $\bigstar$ and $\bigstar\bigstar$, $\lim_{(h,k)\to 0} \frac{|\lambda(h,0)|}{|h|} + \frac{|\lambda(0,k)|}{|k|} = \lim_{(h,0)\to 0} \frac{|\lambda(h,0)|}{|h|} + \lim_{(0,k)\to 0} \frac{|\lambda(0,k)|}{|k|} = 0 + 0 = 0.$

Since $0 \leq \frac{|\lambda(h,k)|}{|(h,k)|} \leq \frac{|\lambda(h,0)|}{|h|} + \frac{|\lambda(0,k)|}{|k|}$, by the squeeze theorem $\lim_{(h,k)\to 0} \frac{|\lambda(h,k)|}{|(h,k)|} = 0.$

Since λ is linear, from the discussion in the lecture we know $\lim_{(h,k)\to 0} \frac{|\lambda(h,k)|}{|(h,k)|} = 0$ implies $\lambda = 0$.

By assumption $g \neq 0$, so there exists $z \in \mathbb{R}^2$ s.t. $z \neq 0$ and $g(\frac{z}{|z|}) \neq 0$. (Note that $\frac{z}{|z|} \in S^1$)

Since λ is the differential of f at $(0,0)$, it must be true that $\lim_{t\to 0} \frac{|f(tz) - f(0,0) - \lambda(tz)|}{|tz|} = 0$, where $t \in \mathbb{R}$. But on the other hand, since $\lambda = 0$ and $g(-\frac{z}{|z|}) = -g(\frac{z}{|z|})$, it follows that

$$\lim_{t\to 0} \frac{|f(tz) - f(0,0) - \lambda(tz)|}{|tz|} = \lim_{t\to 0} \frac{\left| |tz| g(\frac{tz}{|tz|}) \right|}{|tz|} = \lim_{t\to 0} \left| g(\pm \frac{z}{|z|}) \right| = \left| g(\frac{z}{|z|}) \right| \neq 0$$

This is a contradiction. Hence f cannot be differentiable at $(0,0)$.

□

3. *Proof.* Since $|f(x)| \leq |x|$, in particular $|f(0)| \leq |0|^2 = 0$. Thus $f(0) = 0$. Let λ be the zero linear transformation. Then

$$\frac{|f(h) - f(0) - \lambda(h)|}{|h|} = \frac{|f(h)|}{|h|} \leq \frac{|h|^2}{|h|} = |h|$$

Since $\lim_{h \to 0} |h| = 0$ and $\frac{|f(h) - f(0) - \lambda(h)|}{|h|} \geq 0$, by the squeeze theorem we have

$$\lim_{h \to 0} \frac{|f(h) - f(0) - \lambda(h)|}{|h|} = 0.$$

Thus f is differentiable at 0 with derivative $Df(0) = \lambda = 0$. □

4. *(a).* $f(x, y, z) = x^y = e^{y \log x}$, so $f = \exp \circ p \circ g$, where $\exp : \mathbb{R} \to \mathbb{R}$, $p : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$ are defined by $\exp(a) = e^a$, $p(c, d) = c \cdot d$, $p(x, y, z) = (\log x, y)$, respectively.

By the Chain Rule: $Df(x, y, z) = D(\exp \circ p \circ g)(x, y, z) = D\exp(p \circ g(x, y, z)) \cdot Dp(g(x, y, z)) \cdot Dg(x, y, z)$.

$Dg(x, y, z) = \begin{pmatrix} Dg_1(x, y, z) \\ Dg_2(x, y, z) \end{pmatrix}$, where g_1, g_2 are the component functions of g defined by $g_1(x, y, z) = \log x$, $g_2(x, y, z) = y$.

Thus $Dg_1(x, y, z) = \begin{pmatrix} \frac{1}{x} & 0 & 0 \end{pmatrix}$ and $Dg_2(x, y, z) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$

So $Dg(x, y, z) = \begin{pmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. By Spivak Theorem 2-3: $Dp(c, d) = \begin{pmatrix} d & c \end{pmatrix}$.

So $Dp(g(x, y, z)) = Dp(\log x, y) = \begin{pmatrix} y & \log x \end{pmatrix}$.

Since $D\exp(a) = e^a$, $D\exp(p \circ g(x, y, z)) = D\exp(y \log x) = \begin{pmatrix} e^{y \log x} \end{pmatrix} = \begin{pmatrix} x^y \end{pmatrix}$

Therefore, $Df(x, y, z) = \begin{pmatrix} x^y \end{pmatrix} \cdot \begin{pmatrix} y & \log x \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} yx^{y-1} & x^y \log x & 0 \end{pmatrix}$

□

(b). $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z))$ where f_1, f_2 are the component functions of f defined by $f_1(x, y, z) = x^y$, $f_2(x, y, z) = z$. So $Df(x, y, z) = \begin{pmatrix} Df_1(x, y, z) \\ Df_2(x, y, z) \end{pmatrix}$. Clearly $Df_2(x, y, z) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$.

By part(a), we know $Df_2(x, y, z) = \begin{pmatrix} yx^{y-1} & x^y \log x & 0 \end{pmatrix}$. Therefore, $Df(x, y, z) = \begin{pmatrix} yx^{y-1} & x^y \log x & 0 \\ 0 & 0 & 1 \end{pmatrix}$ □

(g). $f(x, y, z) = (x + y)^z = e^{z \log(x+y)}$. So $f = \exp \circ p \circ g$, where $\exp : \mathbb{R} \to \mathbb{R}$ is the exponential function, $p : \mathbb{R}^2 \to \mathbb{R}$ is defined by $p(c, d) = cd$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $g(x, y, z) = (\log(x + y), z)$.

By the Chain Rule, $Df(x, y, z) = D\exp(p \circ g(x, y, z)) \cdot Dp(g(x, y, z)) \cdot Dg(x, y, z)$.

$Dg(x, y, z) = \begin{pmatrix} Dg_1(x, y, z) \\ Dg_2(x, y, z) \end{pmatrix}$ where g_1, g_2 are the component functions of g defined by $g_1(x, y, z) = \log(x + y)$ and $g_2(x, y, z) = z$. Clearly $Dg_2(x, y, z) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$.

On the other hand, $g_1 = \log \circ s$, where $s : \mathbb{R}^3 \to \mathbb{R}$ is defined by $s(x, y, z) = x + y$ and $\log : \mathbb{R}^+ \to \mathbb{R}$.

Apply the Chain Rule again: $Dg_1(x, y, z) = D\log(s(x, y, z)) \cdot Ds(x, y, z)$. Since s is linear, $D_s(x, y, z)$ is simply the matrix of s : $\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$. Clearly $D\log(t) = \frac{1}{t}$, so $D\log(s(x, y, z)) = \begin{pmatrix} \frac{1}{x+y} \end{pmatrix}$.

Thus $Dg_1(x, y, z) = \begin{pmatrix} \frac{1}{x+y} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{x+y} & \frac{1}{x+y} & 0 \end{pmatrix}$

So $Dg(x, y, z) = \begin{pmatrix} Dg_1(x, y, z) \\ Dg_2(x, y, z) \end{pmatrix} = \begin{pmatrix} \frac{1}{x+y} & \frac{1}{x+y} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

By Spivak Theorem 2-3: $Dp(c, d) = \begin{pmatrix} d & c \end{pmatrix}$. So $Dp(g(x, y, z)) = Dp(\log(x + y), z) = \begin{pmatrix} z & \log(x + y) \end{pmatrix}$. Since $D\exp(a) = e^a$, $D\exp(p \circ g(x, y, z)) = D\exp(z\log(x + y)) = \begin{pmatrix} e^{z\log(x+y)} \end{pmatrix} = \begin{pmatrix} (x + y)^z \end{pmatrix}$

Therefore,

$$\begin{aligned} Df(x, y, z) &= \begin{pmatrix} (x + y)^z \end{pmatrix} \cdot \begin{pmatrix} z, \log(x + y) \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{x+y} & \frac{1}{x+y} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} z(x + y)^{z-1} & z(x + y)^{z-1} & (x + y)^z \log(x + y) \end{pmatrix} \end{aligned}$$

□

5. *Proof of (a).* Let $\{e_1, ..., e_n\}$ be the standard basis for $\mathbb{R}^n$ and $\{d_1, ..., d_m\}$ be the standard basis for $\mathbb{R}^m$. Let $h = (h_1, ..., h_n) \in \mathbb{R}^n$ and $k = (k_1, ..., k_m) \in \mathbb{R}^m$ be nonzero. Then since f is bilinear, we have:

$$f(h, k) = f(\sum_{i=1}^{n} h_i e_i, \sum_{j=1}^{m} k_j d_j) = \sum_{i=1}^{n} h_i f(e_i, \sum_{j=1}^{m} k_j d_j) = \sum_{i=1}^{n} h_i \sum_{j=1}^{m} k_j f(e_i, d_j) = \sum_{i,j} h_i k_j f(e_i, d_j)$$

Hence

$$\frac{|f(h, k)|}{|(h, k)|} = \frac{\left|\sum_{i,j} h_i k_j f(e_i, d_j)\right|}{|(h, k)|} \leq \frac{\sum_{i,j} |h_i k_j f(e_i, d_j)|}{|(h, k)|} = \frac{\sum_{i,j} |h_i k_j| \cdot |f(e_i, d_j)|}{|(h, k)|} \qquad \bigstar$$

Let $M_h = max\{|h_i| : 1 \leq i \leq n\}$ and $M_k = max\{|k_j| : 1 \leq j \leq m\}$. Since h, k are nonzero, both M_h, M_k are nonzero. Clearly both M_k, M_h are less than $|(h, k)|$. Thus for any $1 \leq i \leq n, 1 \leq j \leq m$ we have

$$\frac{|h_i k_j| \cdot |f(e_i, d_j)|}{|(h, k)|} = \frac{|h_i| \cdot |k_j| \cdot |f(e_i, d_j)|}{|(h, k)|} \leq \frac{M_h \cdot M_k \cdot |f(e_i, d_j)|}{|(h, k)|} \leq \frac{M_h \cdot M_k \cdot |f(e_i, d_j)|}{M_h} = |M_k| \cdot |f(e_i, d_j)|.$$

Because $\lim_{k \to 0} M_k = \lim_{k \to 0} max\{|k_j| : 1 \leq j \leq m\} = 0$, it follows that $\lim_{k \to 0} |M_k| \cdot |f(e_i, d_j)| = 0$.

Since $0 \leq \frac{|h_i k_j| \cdot |f(e_i, d_j)|}{|(h, k)|} \leq |M_k| \cdot |f(e_i, d_j)|$, by the squeeze theorem $\lim_{(h,k) \to 0} \frac{|h_i k_j| \cdot |f(e_i, d_j)|}{|(h, k)|} = 0$

This is true for all i, j. Thus $\lim_{(h,k) \to 0} \frac{\sum_{i,j} |h_i k_j| \cdot |f(e_i, d_j)|}{|(h, k)|} = 0$.

By ★: $0 \le \frac{|f(h,k)|}{|(h,k)|} \le \frac{\sum_{i,j} |h_i k_j| \cdot |f(e_i, d_j)|}{|(h,k)|}$. So applying the squeeze theorem yields $\lim_{(h,k)\to 0} \frac{|f(h,k)|}{|(h,k)|} = 0$.

□

Proof of (b). Fix $(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$. Define $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ by $\varphi(x,y) = f(a,y) + f(x,b)$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. By the biliearity of f, for $\alpha \in \mathbb{R}, (z,w) \in \mathbb{R}^n \times \mathbb{R}^m$ we have:

$$\begin{aligned}\varphi(\alpha \cdot (z,w) + (x,y)) &= \varphi(\alpha z + x, \alpha w + y) = f(a, \alpha w + y) + f(\alpha z + x, b) \\ &= \alpha \cdot f(a,w) + f(a,y) + \alpha \cdot f(z,b) + f(x,b) \\ &= \varphi(x,y) + \alpha \cdot \varphi(z,w)\end{aligned}$$

Thus φ is a linear map. Consider

$$\frac{|f(a+x,b+y) - f(a,b) - \varphi(x,y)|}{|(x,y)|} = \frac{|f(a,b) + f(a,y) + f(x,b) + f(x,y) - f(a,b) - f(a,y) - f(x,b)|}{|(x,y)|} = \frac{|f(x,y)|}{|(x,y)|}$$

By part(a),

$$\lim_{(x,y)\to 0} \frac{|f(a+x,b+y) - f(a,b) - \varphi(x,y)|}{|(x,y)|} = \lim_{(x,y)\to 0} \frac{|f(x,y)|}{|(x,y)|} = 0$$

By the definition of differentiability and uniqueness of the derivative, φ is precisely $Df(a,b)$.

i.e., $Df(a,b)(x,y) = \varphi(x,y) = f(a,y) + f(x,b)$

□

Proof of (c). Let $p : \mathbb{R}^2 \to \mathbb{R}$ be the product function defined by $p(a,b) = ab$. Let $u, w \in \mathbb{R}$.

Then $p(u \cdot w + a, b) = (uw + a)b = uwb + ab = u \cdot p(w,b) + p(a,b)$.

Similarly $p(a, u \cdot w + b) = a(uw + b) = auw + ab = u \cdot p(a,w) + p(a,b)$. So p is bilinear.

By part(b), $Dp(a,b)(x,y) = p(a,y) + p(x,b) = ay + xb$. But on the other hand, $ay + xb = \begin{pmatrix} b & a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

So by uniqueness of the derivative, $Dp(a,b) = \begin{pmatrix} b & a \end{pmatrix}$, conforming with Spivak Theorem 2-3.

So this is a special case of part(b). □

6. *Proof.* Since $det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, $f : \mathbb{R}^4 \to \mathbb{R}$ is defined by $f(a, b, c, d) = ad - bc$.

We may also regard f as a function from $\mathbb{R}^2 \times \mathbb{R}^2$ into $\mathbb{R}$ defined by $f((a, b),(c, d)) = ad - bc$.

Let $(s, t) \in \mathbb{R}^2$ and $\beta \in \mathbb{R}$. Then

$$\begin{aligned} f(\beta \cdot (s,t) + (a,b), (c,d)) &= f((\beta s + a, \beta t + b), (c,d)) = (\beta s + a)d - (\beta t + b)c \\ &= (ad - bc) + \beta(sd - tc) = f((a,b),(c,d)) + \beta \cdot f((s,t),(c,d)). \end{aligned}$$

Similarly,

$$\begin{aligned} f((a,b), \beta \cdot (s,t) + (c,d)) &= f((a,b), (\beta s + c, \beta t + d)) = a(\beta t + d) - b(\beta s + c) \\ &= (ad - bc) + \beta \cdot (at - bs) = f((a,b),(c,d)) + \beta \cdot f((a,b),(s,t)). \end{aligned}$$

Thus f is bilinear. By Q5, f is differentiable and for $(a, b, c, d) \in \mathbb{R}^4$, $(x, y),(z, w) \in \mathbb{R}^2$, the derivative satisfy:

$$Df((a,b),(c,d))((x,y),(z,w)) = f((a,b),(z,w)) + f((x,y),(c,d)) = aw - bz + xd - yc = \begin{pmatrix} d & -c & -b & a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

By uniqueness of the derivative, $Df(a, b, c, d) = \begin{pmatrix} d & -c & -b & a \end{pmatrix}$ □