1. (a) Let $x \in \mathbb{R}^n$. If $|x| < 1$, then x is contained in the unit open ball, which is a subset of A_1. If $|x| = 1$, then every open ball centered at x intersects both A_1 and ${A_1}^{\mathsf{C}}$. If $|x| > 1$, then x is contained in the open ball centered at x of radius $|x| - 1$, which is disjoint from A_1. Hence, the interior of A_1 is $\{\, x \in \mathbb{R}^n \mid |x| < 1 \,\}$, the exterior of A_1 is $\{\, x \in \mathbb{R}^n \mid |x| > 1 \,\}$, and the boundary of A_1 is $\{\, x \in \mathbb{R}^n \mid |x| = 1 \,\}$.

   (b) Let $x \in \mathbb{R}^n$. If $|x| < 1$, then x is contained in the unit open ball, which is disjoint from A_2. If $|x| = 1$, then every open ball centered at x intersects both A_2 and ${A_2}^{\mathsf{C}}$. If $|x| > 1$, then x is contained in the open ball centered at x of radius $|x| - 1$, which is disjoint from A_2. Hence, the interior of A_2 is $\varnothing$, the exterior of A_2 is $\{\, x \in \mathbb{R}^n \mid |x| \neq 1 \,\}$, and the boundary of A_2 is itself.

   (c) Every open ball in $\mathbb{R}^n$ intersects both A_3 and ${A_3}^{\mathsf{C}}$, so the boundary of A_3 is $\mathbb{R}^n$ and the interior and exterior are $\varnothing$.

2. (a) A is closed, so A^{C} is open, and there exists an open ball $B_r(p) = \{\, q \in \mathbb{R}^n \mid |q - p| < r \,\}$ such that $x \in B_r(p)$ and $B_r(p) \subset A^{\mathsf{C}}$. Let $d = r - |x - p|$. Then $d > 0$, and for all $y \in A$,

$$\begin{aligned} |y - x| &\geq |y - p| - |x - p| \quad \text{(The triangle inequality)} \\ &\geq r - |x - p| \quad (y \notin B_r(p)) \\ &= d. \end{aligned}$$

   (b) By (a), for every $y \in B$, there exsits $d_y > 0$ such that $|y - x| \geq d_y$ for all $x \in A$. The collection of open balls $\left\{\, B_{d_y/2}(y) \,\middle|\, y \in B \,\right\}$ covers B, so by compactness there exist $y_1, \ldots, y_m \in B$ ($m \in \mathbb{Z}_+$) such that $\left\{\, B_{d_{y_1}/2}(y_1), \ldots, B_{d_{y_m}/2}(y_m) \,\right\}$ covers B. Let $d = \min_{i=1}^{m}\{d_{y_i}/2\} \in \mathbb{R}_+$. Let $y \in B$. Then there is some i such that $y \in B_{d_{y_i}/2}(y_i)$. For all $x \in A$,

$$\begin{aligned} |y - x| &\geq |y_i - x| - |y - y_i| \\ &> d_{y_i} - d_{y_i}/2 \\ &= d_{y_i}/2 \\ &\geq d. \end{aligned}$$

   (c) Let $A = \mathbb{R} \times \{\, 0 \,\}$, $B = \{\, (x, 1/x) \mid x \in \mathbb{R}_+ \,\}$. A and B are closed; also, they are unbounded and thus not compact. For every $d > 0$, if $x > 1/d$ then the points $(x, 0)$ and $(x, 1/x)$ are in A and B respectively, but their distance is $1/x < d$.

3. If $C = \varnothing$, we can take $D = \varnothing$. Assume $C \neq \varnothing$.

   Since U^{C} is closed and $C \subset U$ is compact, it follows from Q2 (b) that there is some $d > 0$ such that $|y - x| \geq d$ for all $x \in U^{\mathsf{C}}$ and $y \in C$, i.e. $B_d(y) \subset U$ for all $y \in C$.

   Since C is compact and the collection of open balls $\left\{\, B_{d/2}(y) \,\middle|\, y \in C \,\right\}$ covers C, there exist $y_1, \ldots, y_m \in C$ ($m \in \mathbb{Z}_+$) such that $\left\{\, B_{d/2}(y_1), \ldots, B_{d/2}(y_m) \,\right\}$ covers C. Let $D = \bigcup_{i=1}^{m} \bar{B}_{d/2}(y_i)$. (Let $\bar{B}_r(p)$, where $p \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, denote the closed ball centered at p with radius r.)

   - D is closed because it is a finite union of closed sets.

     By compactness, $|y| \leq M$ for all $y \in C$ for some $M \in \mathbb{R}_+$. Let $p \in D$. Then there is some i such that $p \in \bar{B}_{d/2}(y_i)$, so $|p| \leq |p - y_i| + |y_i| \leq d/2 + M$. Therefore, D is bounded.

     Since D is closed and bounded, it follows that D is compact.

- Let $y \in C$. Then there is some i such that $y \in B_{d/2}(y_i)$. Since $B_{d/2}(y_i) \subset D$, it follows that $y \in \operatorname{Int} D$. Hence, $C \subset \operatorname{Int} D$.
- For all i, $\bar{B}_{d/2}(y_i) \subset B_d(y_i) \subset U$, so $D \subset U$.

4. By Assignment 1 Q3, there is some $M \in \mathbb{R}_+$ such that $|Tv| \leq M\,|v|$ for all $v \in \mathbb{R}^n$.

   Let $v \in \mathbb{R}^n$. Let $\epsilon \in \mathbb{R}_+$ be given. Take $\delta = \epsilon/M$. Then for all $u \in \mathbb{R}^n$ such that $|u - v| < \delta$, we have $|Tu - Tv| = |T(u - v)| \leq M\,|u - v| < \epsilon$. Therefore, T is continuous.

5. Let $\epsilon = 1$. Let $\delta \in \mathbb{R}_+$. Define $g : \mathbb{R} \to \mathbb{R}$, $y \mapsto y^2 + y - \delta^2/4$. Since g is continuous and $g(0) = -\delta^2/4 < 0$, there is some $y_0 > 0$ such that $g(y_0) < 0$. Then $\delta^2/4 - {y_0}^2 > y_0 > 0$. Let $x_0 = \sqrt{\delta^2/4 - {y_0}^2}$. Then $|(x_0, y_0)| = \delta/2$, $x_0 > 0$, and $y_0 < \delta^2/4 - {y_0}^2 = {x_0}^2$. Hence, $|(x_0, y_0) - (0, 0)| < \delta$ but $|f(x_0, y_0) - f(0, 0)| = 1 = \epsilon$. This implies that f is not continuous at $(0, 0)$.

   Let L be a straight line through $(0, 0)$.

   - If L is the y-axis or the slope of L is nonpositive, then $y \leq 0$ for all $(x, y) \in L$ such that $x > 0$, so $f|_L = 0$ and thus $f|_L$ is continuous at $(0, 0)$.
   - Let $k \in \mathbb{R}_+$ be the slope of L. Then for $(x, y) \in L$ such that $|(x, y) - (0, 0)| < k\sqrt{1 + k^2}$,
     - if $x \leq 0$, then surely $f(x, y) = 0$;
     - if $x > 0$, then since $\sqrt{x^2 + (kx)^2} < k\sqrt{1 + k^2}$, we have $0 < x < k$, so $y = kx > x^2$, which implies $f(x, y) = 0$.

     Hence, $f|_L(x, y) = 0$ for all $(x, y) \in L$ such that $|(x, y) - (0, 0)| < k\sqrt{1 + k^2}$, so $f|_L$ is continuous at $(0, 0)$.

6. Since A is not closed, there is some $x_0 \in A^{\complement}$ such that every open ball centered at x_0 intersects A. For all $x \in A$, define $f(x) = 1/\,|x - x_0|$.

   f is the composition of the function $A \to \mathbb{R}_+$, $x \mapsto |x - x_0|$ and the continuous function $\mathbb{R}_+ \to \mathbb{R}$, $x \mapsto 1/x$. Let $\epsilon \in \mathbb{R}_+$ be given. Take $\delta = \epsilon$. Then for all $x, x' \in A$, if $|x' - x| < \delta$, then $||x' - x_0| - |x - x_0|| \leq |x' - x| < \epsilon$, so the former is also continuous. Hence, f is continuous.

   Let $M \in \mathbb{R}_+$ be given. By our choice of x_0, the open ball $B_{1/M}(x_0)$ intersects A, so there is some $x \in A$ such that $|x - x_0| < 1/M$ and thus $f(x) > M$. Hence, f is unbounded.