Q1. (**Spivak**'s 1-1)

Prove that $||x|| \leq \sum_{i=1}^{n} |x_i|$.

Proof: First, claim that $\sqrt{\sum_{i=1}^{n} a_i} \leq \sum_{i=1}^{n} \sqrt{a_i}$ assuming all $a_i \geq 0$.

Proof of the claim by induction:

Base case: $n = 1$, the claim becomes an equality,

there is nothing to prove.

$$n = 2,\ 0 \leq \sqrt{a_1 a_2} \implies a_1 + a_2 \leq a_1 + a_2 + 2\sqrt{a_1 a_2}$$
$$\implies \sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2}$$

Induction step: Assume that the claim holds for some $n \in \mathbb{N}$,

WTS it still holds for $n + 1$.

$$\begin{aligned}
\sqrt{\sum_{i=1}^{n+1} a_i} &= \sqrt{\Big(\sum_{i=1}^{n} a_i\Big) + a_{n+1}} \\
&\leq \sqrt{\sum_{i=1}^{n} a_i} + \sqrt{a_{n+1}} \qquad (1) \\
&\leq \Big(\sum_{i=1}^{n} \sqrt{a_i}\Big) + \sqrt{a_{n+1}} \quad (2) \\
&= \sum_{i=1}^{n+1} \sqrt{a_i}
\end{aligned}$$

(1) is true by base case of $n = 2$,

(2) is true by the induction hypothesis.

This completes the proof of the claim.

Back to the question, $||x|| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{n} {x_i}^2} \leq \sum_{i=1}^{n} \sqrt{{x_i}^2} = \sum_{i=1}^{n} |x_i|$.

$\square$

Q2. (**Spivak**'s 1-7)

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is norm preserving if $||T(x)|| = ||x||$ for all x, and inner product preserving if $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all x, y.

(a) Prove that T is norm preserving iff it is inner product preserving.

(b) Prove that such a linear transformation is 1-1 and onto, and that T^{-1} is also norm and inner product preserving.

(a) Proof: "$\Longleftarrow$": Assume T is inner product preserving.

Then take $y = x$, we have $\langle T(x), T(x)\rangle = \langle x, x\rangle$ for all x.

This means that $||T(x)|| = ||x||$ for all x.

Hence, T is norm preserving.

"$\Longrightarrow$": Assume T is norm preserving.

Using the fact that T is linear, pick any x and y,

we have $||T(x) + T(y)|| = ||T(x + y)|| = ||x + y||$

$$\Longrightarrow \langle T(x) + T(y), T(x) + T(y)\rangle = \langle x + y, x + y\rangle$$

$$\Longrightarrow \langle T(x), T(x)\rangle + \langle T(y), T(y)\rangle + 2\langle T(x), T(y)\rangle$$

$$= \langle x, x\rangle + \langle y, y\rangle + 2\langle x, y\rangle \qquad \text{by F.O.I.L rule}$$

$$\Longrightarrow \langle T(x), T(y)\rangle = \langle x, y\rangle, \quad \text{since } ||T(x)|| = ||x|| \ \forall x$$

Hence, T is inner product preserving.

□

(b) Proof: 1-1: Take $T(x) = T(y)$. Then WTS $x = y$.

$$\begin{aligned} T(x) = T(y) &\implies T(x-y) = 0 \\ &\implies ||T(x-y)|| = ||x-y|| = 0 \\ &\implies x - y = 0 \implies x = y \end{aligned}$$

onto: T is 1-1, then $\text{null}(T) = \{0\}$.

So by rank-nullity theorem:

$$\begin{aligned} &\dim(\text{range}(T)) \\ &= \dim(\text{range}(T)) + \dim(\text{null}(T)) \\ &= \dim(\text{Domain}) = \dim(\text{Codomain}) \end{aligned}$$

This means that T is onto.

T^{-1}: T is 1-1 and onto, so T^{-1} is well-defined.

Given $T^{-1}(x)$, we have $||T(T^{-1}(x))|| = ||T^{-1}(x)|| = ||x||$.

This shows the inverse is norm preserving.

Then it is also inner product preserving by (a).

□

Q3. (**Spivak**'s 1-10)

If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, show that there is a number M such that $||T(h)|| \leq M||h||$ for all $h \in \mathbb{R}^m$.

Proof: Write $M_T = \begin{pmatrix} a_{1,1} & \cdots\cdots & a_{1,m} \\ \vdots & & \\ a_{n,1} & \cdots\cdots & a_{n,m} \end{pmatrix}$, and $h = (h_1, \ldots, h_m)$.

Then, $T(h) = (b_1, \ldots, b_n)$, where $b_i = \sum_{j=1}^{m} a_{i,j} h_j$.

Let $M = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} {a_{i,j}}^2}$

$$
\begin{aligned}
\text{Then}||T(h)||^2 &= \sum_{i=1}^{n} {b_i}^2 \\
&= \sum_{i=1}^{n} \Big(\sum_{j=1}^{m} a_{i,j} h_j \Big)^2 \\
&\leq \sum_{i=1}^{n} \Big(\sum_{j=1}^{m} {a_{i,j}}^2 \cdot \sum_{j=1}^{m} {h_j}^2 \Big) \qquad \bigstar \\
&= \sum_{j=1}^{m} {h_j}^2 \cdot \sum_{i=1}^{n} \sum_{j=1}^{m} {a_{i,j}}^2 \\
&= M^2 ||h||^2
\end{aligned}
$$

$\bigstar$ is true because of Theorem 1-1 (2) (**Spivak**, Page 2):

$$\Big| \sum_{i=1}^{n} x_i \cdot y_i \Big| \leq ||x|| \cdot ||y||,$$

which can be written as $\Big(\sum_{i=1}^{n} x_i \cdot y_i \Big)^2 \leq \sum_{i=1}^{n} {x_i}^2 \cdot \sum_{i=1}^{n} {y_i}^2$

□

Q3. (**Spivak**'s 1-10) (Alterantive Solution)

If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, show that there is a number M such that $||T(h)|| \leq M||h||$ for all $h \in \mathbb{R}^m$.

Proof: Write $M_T = \begin{pmatrix} a_{1,1} & \cdots\cdots & a_{1,m} \\ \vdots & & \\ a_{n,1} & \cdots\cdots & a_{n,m} \end{pmatrix}$, and $h = (h_1, \ldots, h_m)$.

Then, $T(h) = (b_1, \ldots, b_n)$, where $b_i = \sum_{j=1}^{m} a_{i,j} h_j$.

Let $L = \max\{|a_{i,j}|\}$ for all i, j.

$$
\begin{aligned}
\text{Then}||T(h)||^2 &\leq \Big(\sum_{i=1}^{n} |b_i|\Big)^2 && (1) \\
&= \Big(\sum_{i=1}^{n} \Big|\sum_{j=1}^{m} a_{i,j} h_j\Big|\Big)^2 \\
&\leq \Big(\sum_{i=1}^{n}\sum_{j=1}^{m} \Big|a_{i,j} h_j\Big|\Big)^2 && \text{by Triangle Inequality} \\
&\leq \Big(\sum_{i=1}^{n}\sum_{j=1}^{m} L\Big|h_j\Big|\Big)^2 \\
&= \Big(nL\sum_{j=1}^{m} \Big|h_j\Big|\Big)^2 \\
&= n^2L^2\Big(\sum_{j=1}^{m} \Big|h_j\Big|\Big)^2 \\
&\leq n^3L^2 \sum_{j=1}^{m} |h_j|^2 && (2) \\
&= n^3L^2||h||^2
\end{aligned}
$$

(1) is true because of Question 1, $||x|| \leq \sum_{i=1}^{n} |x_i|$.

(2) is true because of $\Big|\sum_{i=1}^{n} x_i \cdot y_i\Big| \leq ||x|| \cdot ||y||$

which is equivalent to $\Big(\sum_{i=1}^{n} x_i \cdot y_i\Big)^2 \leq \sum_{i=1}^{n} {x_i}^2 \cdot \sum_{i=1}^{n} {y_i}^2$

which is equivalent to $\Big(\sum_{i=1}^{n} |x_i|\Big)^2 \leq \sum_{i=1}^{n} |x_i|^2 \cdot n$

$\square$

Q4. (**Spivak**'s 1-12)

Let $(\mathbb{R}^n)^*$ denote the dual space of the vector space $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, define $\varphi_x \in (\mathbb{R}^n)^*$ by $\varphi_x(y) = \langle x, y \rangle$. Define $T : \mathbb{R}^n \to (\mathbb{R}^n)^*$ by $T(x) = \varphi_x$. Show that T is a 1-1 linear transformation and conclude that every $\varphi \in (\mathbb{R}^n)^*$ is φ_x for a unique $x \in \mathbb{R}^n$.

Proof: T is a linear transformation:

(i): Given $x, y, z \in \mathbb{R}^n$,

$\varphi_{x+y}(z) = \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle = \varphi_x(z) + \varphi_y(z)$

This is true for all $z \in \mathbb{R}^n$.

So $\varphi_{x+y} = \varphi_x + \varphi_y$, which means $T(x + y) = T(x) + T(y)$.

(ii): Given $x, y \in \mathbb{R}^n, a \in \mathbb{R}$,

$\varphi_{ax}(y) = \langle ax, y \rangle = a\langle x, y \rangle = a \cdot \varphi_x(y)$

This is true for all $y \in \mathbb{R}^n$.

So $\varphi_{ax} = a\varphi_x$, which means $T(ax) = aT(x)$.

(i) + (ii) $\implies$ T is a linear transformation.

T is 1-1:

We will prove T is 1-1 by proving $\text{null}(T) = \{0\}$.

Suppose $x \in \mathbb{R}^n$ and $T(x) = \varphi_x = 0$.

Then $\varphi_x(y) = 0$ for all $y \in \mathbb{R}^n$

Take $y = x$, we have $\varphi_x(x) = \langle x, x \rangle = ||x||^2 = 0$

This means that $x = 0$, hence $\text{null}(T) = \{0\}$.

Every $\varphi \in (\mathbb{R}^n)^*$ is φ_x for a unique $x \in \mathbb{R}^n$:

This is equivalent to show T is bijection,

So we need to show T is onto i.e. $\dim(\text{range}(T)) = \dim((\mathbb{R}^n)^*)$.

By rank-nullity theorem and $\dim((\mathbb{R}^n)^*) = \dim(\mathbb{R}^n)$,

$$\begin{aligned}\dim(\text{range}(T)) &= \dim(\text{range}(T)) + \dim(\text{null}(T)) \\ &= \dim(\mathbb{R}^n) = \dim((\mathbb{R}^n)^*).\end{aligned}$$ $\square$

Q5. (**Spivak**'s 1-13)

If $x, y \in \mathbb{R}^n$, then x and y are called perpendicular if $\langle x, y \rangle = 0$. If x and y are perpendicular, show that $||x+y||^2 = ||x||^2 + ||y||^2$.

Proof: x and y are called perpendicular $\implies \langle x, y \rangle = 0$

$$\begin{aligned} ||x+y||^2 &= \langle x+y, x+y \rangle \\ &= \langle x, x \rangle + \langle y, y \rangle + 2\langle x, y \rangle, \text{ by F.O.I.L rule} \\ &= ||x||^2 + ||y||^2 \end{aligned}$$

□

Q6. (**Spivak**'s 1-18)

If $A \subseteq [0,1]$ contains all the rational numbers in $(0,1)$ and is the union of open intervals (a_i, b_i), show that the boundary of A is $[0,1] \setminus A$.

Proof: We will show $\text{bd}(A) \subseteq [0,1] \setminus A$ and $[0,1] \setminus A \subseteq \text{bd}(A)$.

(1): Pick $p \in \text{bd}(A)$.

So any open neighbourhood of p contains a point not in A.

Then $A = \bigcup(a_i, b_i) \implies A$ is open $\implies p \notin A$.

And if $p \notin [0,1]$, then there exists an open neighbourhood,

$$U_p \subseteq \mathbb{R} \setminus [0,1].$$

$U_p \cap A = \varnothing$. This contradicts the definition of boundary.

Hence, $p \in [0,1]$.

$p \in [0,1]$ and $p \notin A \implies p \in [0,1] \setminus A$

This means $\text{bd}(A) \subseteq [0,1] \setminus A$.

(2): Pick $p \in (0,1) \setminus A$ and any open neighbourhood U_p of p.

$p \in (0,1) \implies U_p \cap (0,1) \neq \varnothing$, where the LHS is a union of two open sets, hence itself is open.

$\implies$ there exists a rational number $a \in U_p \cap (0,1)$ since $\mathbb{Q}$ is dense.

a is rational and is in $(0,1) \implies a \in A$.

So $a \in U_p \cap A \neq \varnothing$, and $p \in U_p \cap (\mathbb{R} \setminus A) \neq \varnothing$.

Hence, $p \in \text{bd}(A)$.

This means $(0,1) \setminus A \subseteq \text{bd}(A)$.

And clearly $\{0,1\} \subseteq \text{bd}(A)$ since both have open neighbourhoods that have a point in A (a rational larger than 0 or smaller than 1) and a point not in A (any point larger than 1 or smaller than 0).

So we can further have $[0,1] \setminus A \subseteq \text{bd}(A)$.

(1) + (2) $\implies [0,1] \setminus A = \text{bd}(A)$ □

Q7. (**Spivak**'s 1-19)

If A is closed set that contains every rational number in $[0,1]$, show that $[0,1] \subseteq A$.

Proof: Pick any point $p \in (0,1)$ and any open neighbourhood U_p of p.

We will prove $p \in A$ by contradiction. Assume $p \notin A$.

$p \in (0,1) \implies U_p \cap (0,1) \neq \varnothing$, where the LHS is a union of two open sets, hence itself is open.

$\implies$ there exists a rational number $a \in U_p \cap (0,1)$

since $\mathbb{Q}$ is dense.

a is rational and is in $(0,1) \implies a \in A$.

So $a \in U_p \cap A \neq \varnothing$

$p \notin A \implies p \in \mathbb{R} \setminus A$, which is an open set.

This is a contradiction, since any open neighbourhood of p contains a point in A, i.e. not in $\mathbb{R} \setminus A$.

Hence, $p \in A$. So $(0,1) \subseteq A$.

And clearly $\{0,1\} \subseteq A$ since they are rational numbers in $[0,1]$.

So we can further have $[0,1] \subseteq A$. $\square$