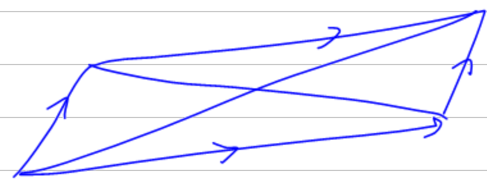
**Def** $\langle x, y\rangle = \frac{1}{4}\left(|x+y|^2 - |x-y|^2\right)$

$$\langle x+y, z\rangle \overset{?}{=} \langle x, z\rangle + \langle y, z\rangle$$

$$|x+y|^2 + |x-y|^2 = 2\left(|x|^2 + |y|^2\right)$$

not inside

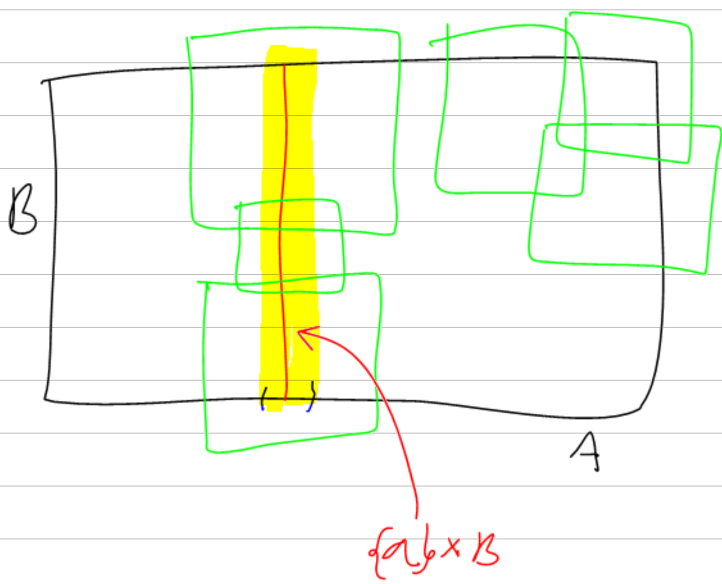


$a+b = x+y+z$ $\qquad a = x + \frac{1}{2}y$

$a-b = x-z$ $\qquad b = z + \frac{1}{2}y$

circle: $\{x : |x-a| = r\}$

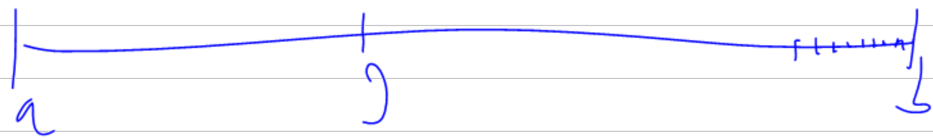
$a \in \mathbb{R}^2 \quad r > 0$




3.14159265358979323846264338327950288419716939937510582097494459230 7

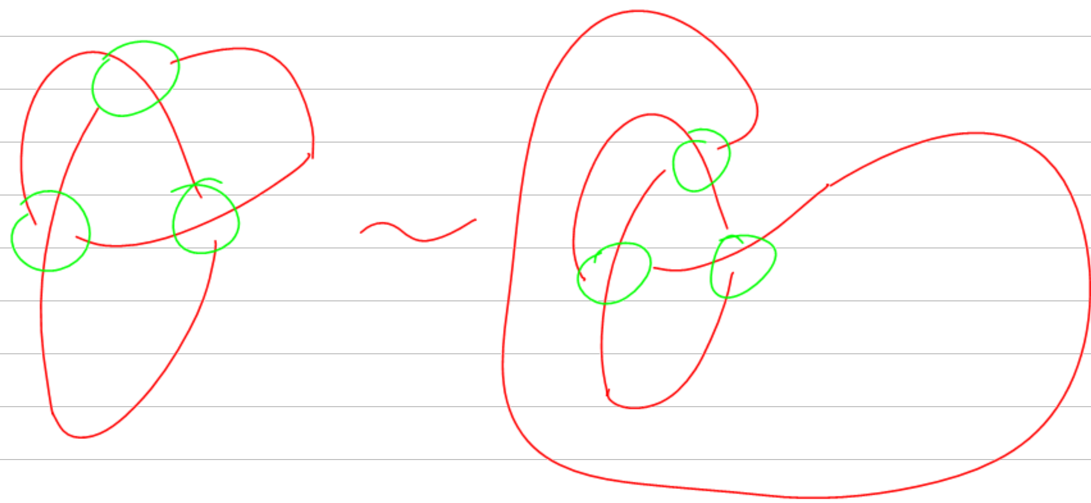
2 2
5 5
7 7

$G = \{ g \in [a,b] : [a,g]$ can be covered by finitely many elements of $\mathcal{U} \}$

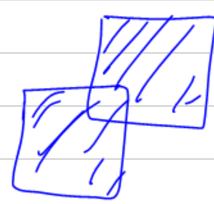



$\gamma = \sup G$

1. $\gamma > a$
2. $\gamma = b$
3. $b \in G$.

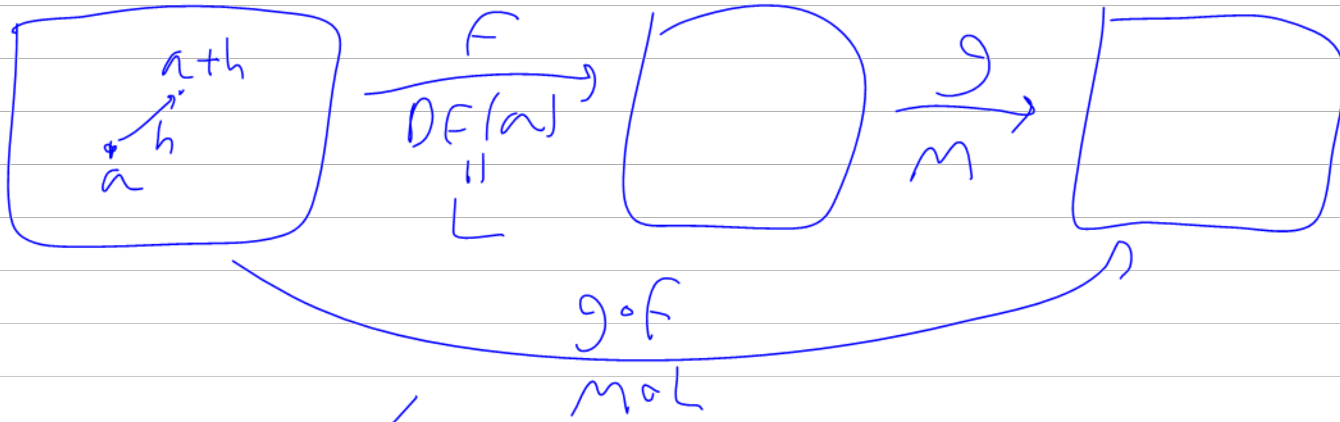


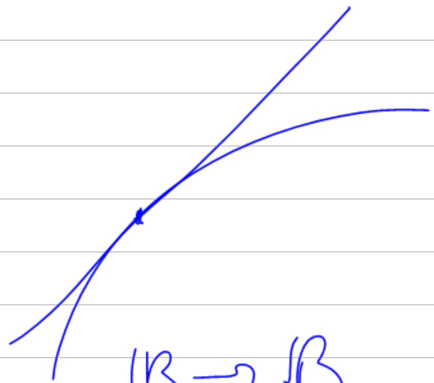
$A_{1,2}$ $B_{1,2}$



$A_1 \times B_1 \cup A_2 \times B_2$

$\mathbb{R}^2 \subset \mathbb{R}^3$



$\mathbb{R} \to \mathbb{R}$

$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$

$Df = L$

$f : \mathbb{R} \to \mathbb{R}$

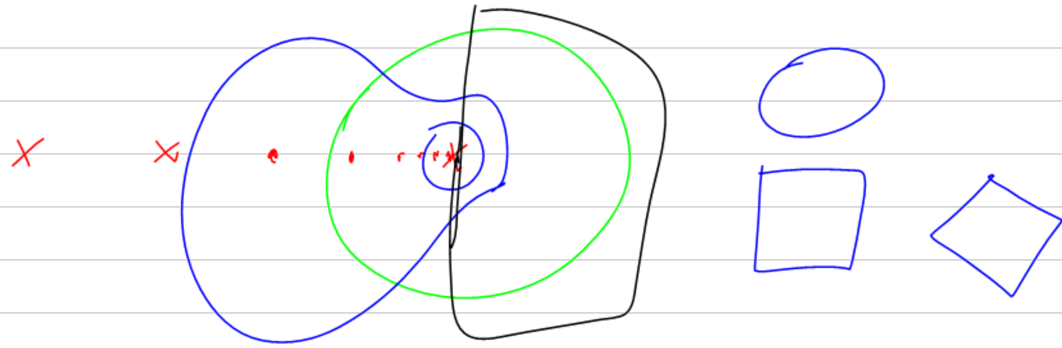
$$f(a+h) = f(a) + f'(a) \cdot h + e(h)$$

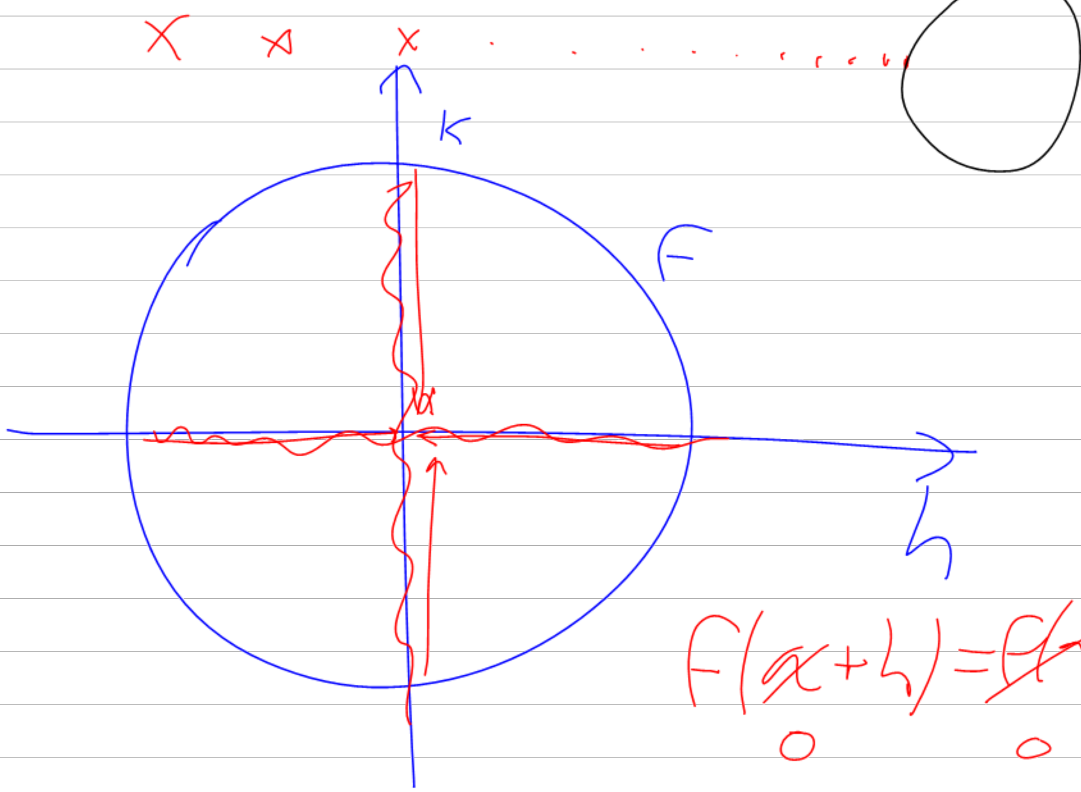
$$\frac{f}{g} \rightsquigarrow \frac{f'}{g'} \qquad \frac{Df(a)}{Dg(a)}$$

$f : \mathbb{R} \to \mathbb{R}$

$$\forall x \ \forall \epsilon \ \exists \delta \ \forall y \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

$$\forall \epsilon \ \exists \delta \ \forall x \ |x-y| < \delta \Rightarrow \_ \ \_ \ \_ .$$






$$F(\underset{0}{\cancel{a}}+h) = \underset{0}{\cancel{F(a)}} + DF(a)\cdot h + o(h)$$

$$0 = DF(0)\cdot\begin{pmatrix} h \\ k \end{pmatrix} + o\begin{pmatrix} h \\ k \end{pmatrix}$$

$$f(a+h) = f(a) + \overset{DF(a)}{L}\cdot h + e(h)$$

$$e(h) = f(a+h) - f(a) - \underline{\underline{L}}\cdot h$$

$$F:\underset{a}{\mathbb{R}^n}\times\underset{b}{\mathbb{R}^m}\longrightarrow\mathbb{R}^p$$

$$(a,b)\in\mathbb{R}^n\times\mathbb{R}^m$$

$$Df(a,b):\underset{x}{\mathbb{R}^n}\times\underset{y}{\mathbb{R}^m}\longrightarrow\mathbb{R}^p$$

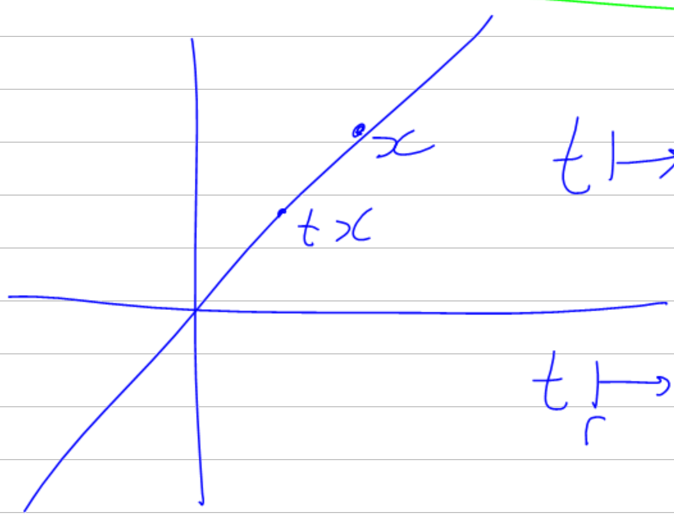
$$Df(a,b)(x,y)\in\mathbb{R}^p$$

$$(x,y)\longmapsto f(x,b)+f(a,y)$$

$$(a,b)+(h,k)=(a+h,b+k)$$

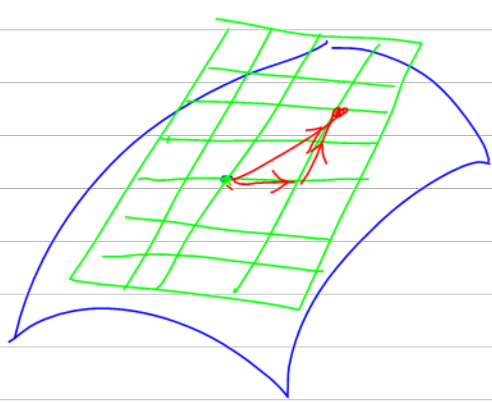
$$f(a+h,b+k)=f(a,b)+\underset{\text{Diff}}{\boxed{\phantom{D}}}\cdot(h,k)+o(\dots)$$

---




$$t\longmapsto f(tx)$$

$$t\longmapsto t\cdot x\longmapsto f(t\cdot x)$$



$$f(a)+L\cdot h$$

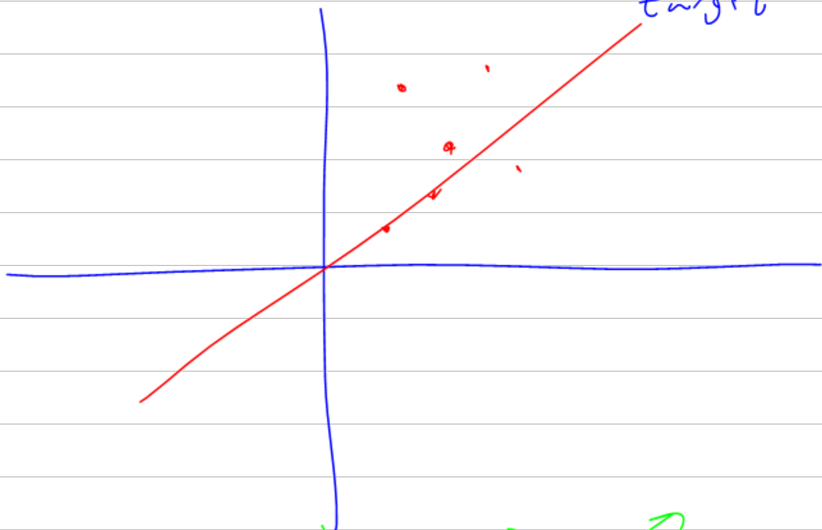
$$\shortparallel$$

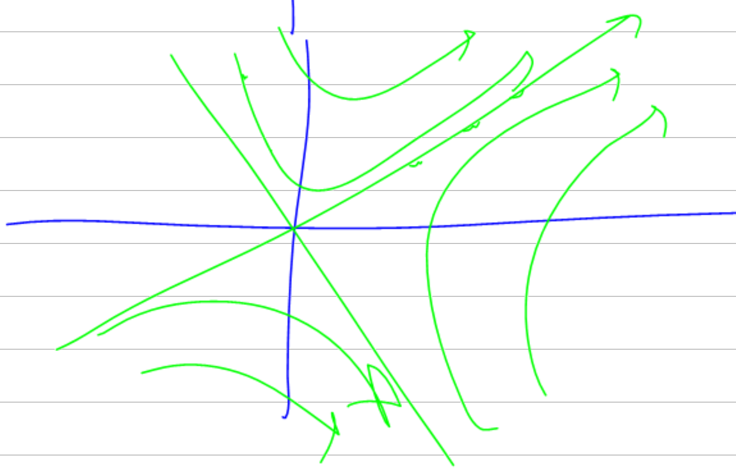
$$D_1f\cdot h_1+D_2f\cdot h_2$$

$D_1(\sin xy) = y\cos(xy)$

$F: \mathbb{R}^n \to \mathbb{R}^m \quad DF(a) \in M_{m\times n}$

$F: \mathbb{R}^n \to \mathbb{R}^n \quad \underbrace{L\cdot V}_{\text{target}} = \underbrace{\lambda V}_{\text{domain}}$

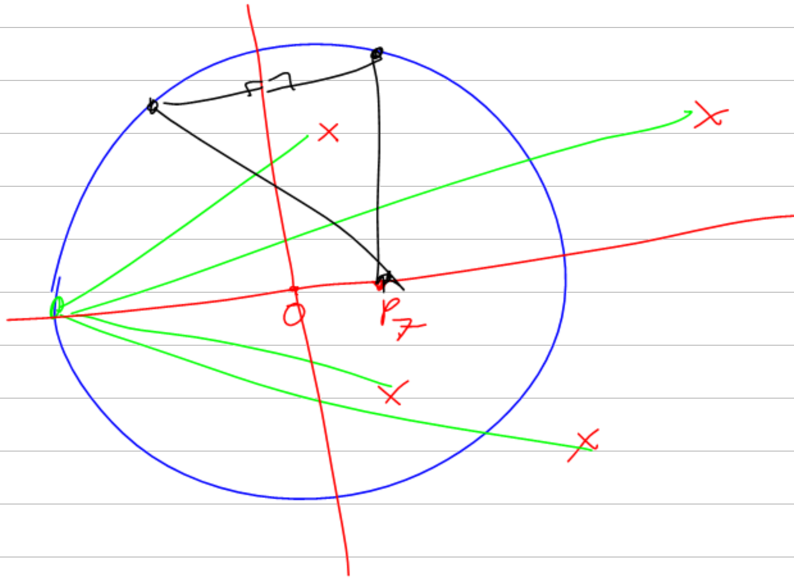




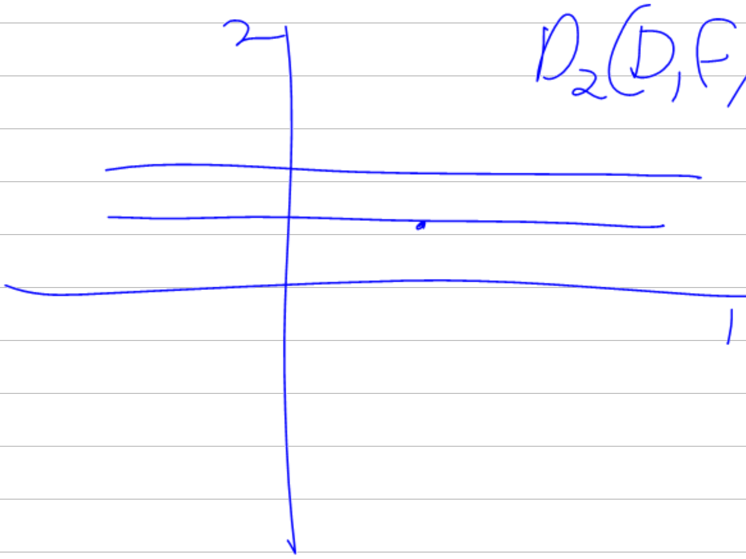
$f(x) = \sum x_i g_i(x)$

$g_2 \dots g_n = 0 \quad g_1(x) = \begin{cases} \frac{f(x)}{x_1} & x_1 \neq 0 \\ 0 & \text{otherwise} \end{cases}$

$x_1 = 0,\ x \neq 0 \quad ($




$D_2(D,F)$

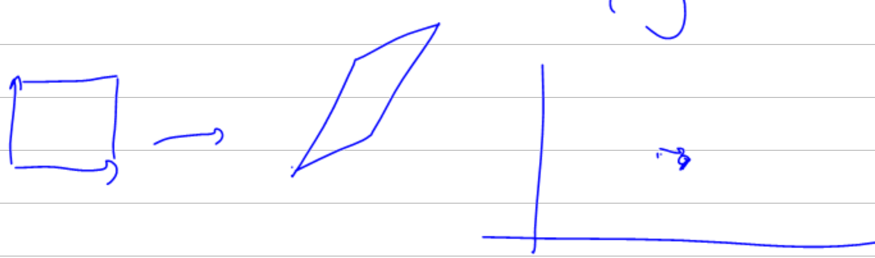



If $F \circ g$ & $g$ is invertible

then $F^{-1} = g \circ (F \circ g)^{-1}$

$\bar{F} = F \circ (\lambda)^{-1}$

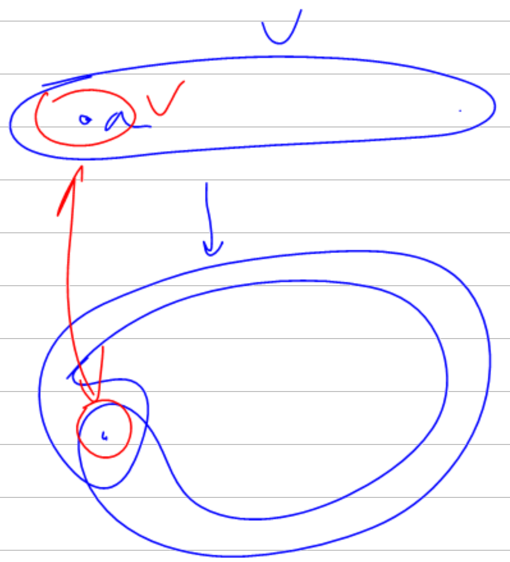
$$\begin{tikzcd} \bullet \arrow[r, "g"] \arrow[rr, bend right, "F \circ g"'] & \bullet \arrow[r, "F"] & \bullet \end{tikzcd}$$



$a_1 \ldots\ldots\ldots\ldots x_n \in \mathbb{R}$

Compute how many increasing subsequences of length 5 are there?

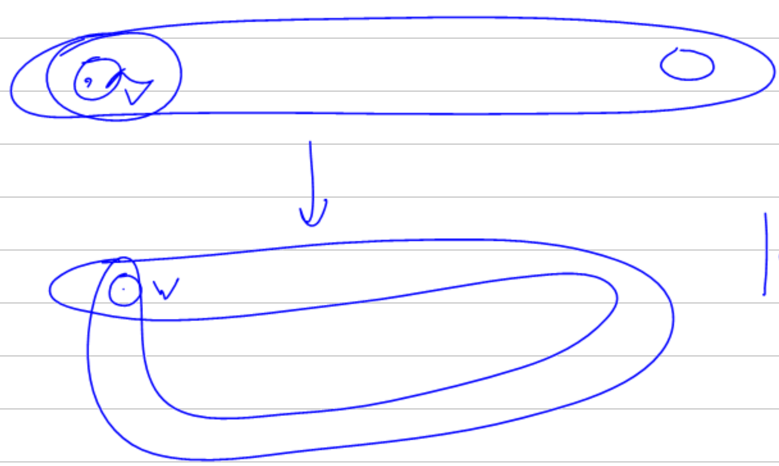
time $\sim n^5$ naively



$A \subset B$

$257^{n^2}$

$257 \cdot n^2$



$$|\alpha-\beta| \geq \big||\alpha|-|\beta|\big| \geq |\alpha|-|\beta|$$

$$\big|(f(x_1)-f(x_2))+(x_1-x_2)\big| < \frac{1}{257}|x_1-x_2|$$

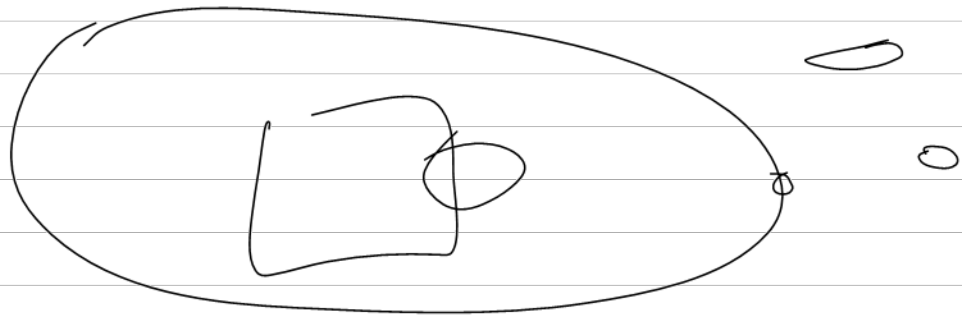
$$\rotatebox{90}{$\leq$}$$

$$-|f(x_1)-f(x_2)|+|x_1-x_2|$$

Weak IFT: Given $F$ w/ $F'(a)$ invertible, $\exists F^{-1}$ ... ~~$F^{-1}$ is cont diffable~~

$F^{-1}$ is diffable at $a$.

Lemma: WIFT = IFT.



$$\begin{pmatrix} 1 & & & & \\ & 1 & & & 0 \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & 0 \end{pmatrix} = L_k$$

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad F(0) = 0$$

rank $F' = k$ near $0$

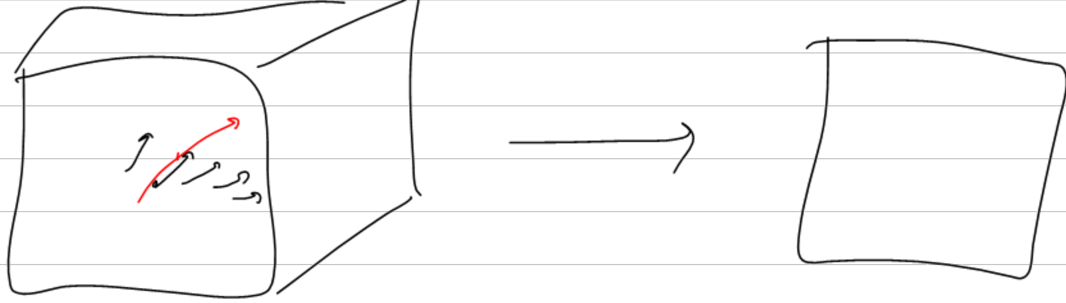
$\exists$ invertible $\phi : \mathbb{R}^n \circlearrowleft$

$\psi : \mathbb{R}^m \circlearrowleft$

s.t. $\psi \circ F \circ \phi = L_k$

$\top$

$(x)$



$F: \mathbb{R}^n \longrightarrow V$ $\qquad \int_R F$

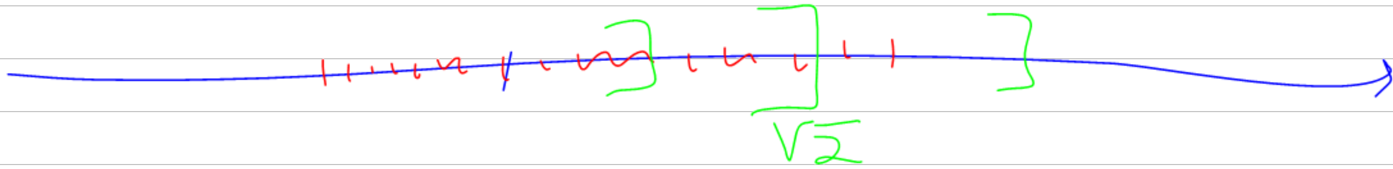
$$F(x, y) = 0 \implies$$
$$F(x, y(x)) = 0$$

$$\frac{\partial}{\partial x} \qquad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} = 0$$

$$\mathbb{R} = \left\{ \begin{matrix} \text{Cauchy} \\ \text{sequences} \\ \text{of} \\ \text{rationals} \end{matrix} \right\} \Big/ \sim$$

$3, 3.1, 3.14, 3.141$

"Dedekind cuts"




$$T: V \longrightarrow W$$

$$T^*: W^* \longrightarrow V^*$$

given $\langle , \rangle : W \longrightarrow V$

$$\mathbb{C} = \mathbb{R}^2$$

$$\mathbb{C}^n = \mathbb{R}^{2n}$$

$F : \mathbb{C} \to \mathbb{C}$ $\quad L : \mathbb{C} \to \mathbb{C}$ $\begin{pmatrix} a & -b \\ +b & a \end{pmatrix}$

$F : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad L : \mathbb{R}^2 \to \mathbb{R}^2$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$F : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ s.t. $F' \in \operatorname{Im} \operatorname{Lin}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m)$

---

$$z \qquad (z) : \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \| & & \| \\ \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \end{array} \longrightarrow \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

$S_1, S_2 : \{\text{edges}\} \to \mathbb{Z}/2$

$S = S_1 + S_2$ is "even around faces"

$\Rightarrow \exists\, \sigma : \{\text{verts}\} \to \mathbb{Z}/2$

s.t. $\underbrace{(S_1 + S_2)(\xi)}_{S(\xi)} = \sigma(\xi_1) - \sigma(\xi_0)$

$$\text{Face} \xrightarrow{\partial} \text{edges} \xrightarrow{\partial} \text{vertices}$$

$$\text{verts} \xrightarrow{d} \text{edges} \xrightarrow{d} \text{faces}$$

$$\sigma \in C^0 \longrightarrow s \in C^1 \longrightarrow ds = 0$$

$$d\sigma = s$$

$$H^1\left(\begin{matrix}\text{2-skeleton}\\ \text{of a cube}\end{matrix}, \mathbb{Z}/2\right)$$

$\frac{\partial f}{\partial y}$ invertible

$\forall x \ \exists y$ s.t. $f(x,y) = 0$

$$f(x,y) = xy \quad (a,b) = (0,0)$$

$$\frac{\partial f}{\partial y}(a,b) = 0$$

$$x^2 + y^2 = 1$$

$$2x + 2yy' = 0$$

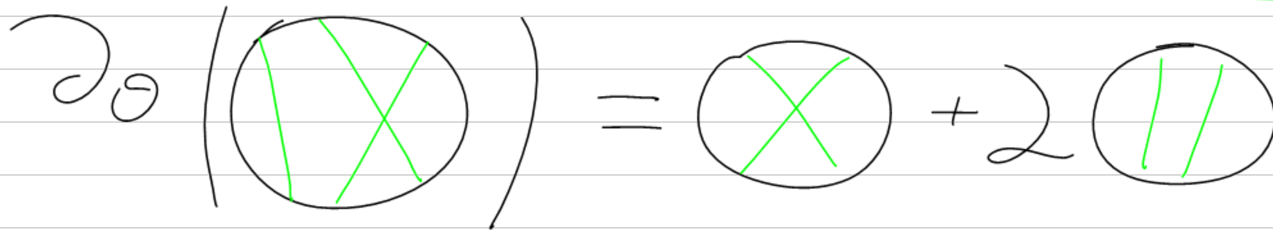
$$y' = -\frac{x}{y} = -\frac{x}{\sqrt{1-x^2}}$$

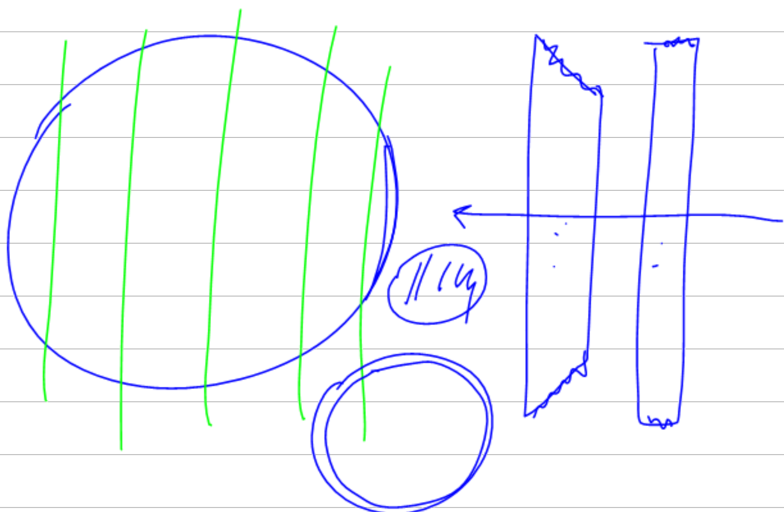
Suppose $g$: s.t. $y(x)$

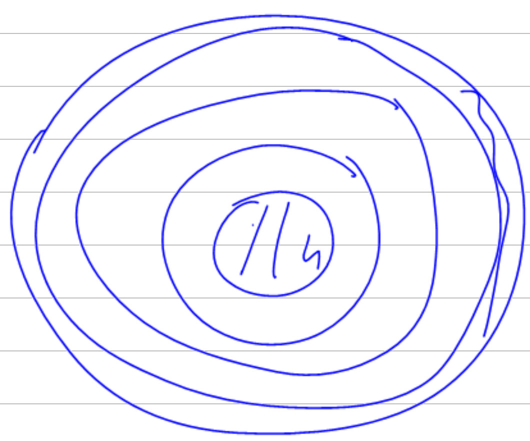
$$f(x, g(x)) = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} = 0$$

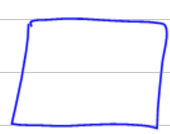
$$F : \underset{x}{\mathbb{R}^n} \times \underset{y}{\mathbb{R}^k} \to \mathbb{R}^{k+1}$$

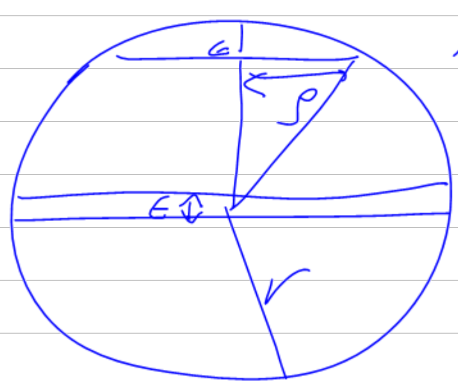






$$\int_M d\omega = \int_{\partial M} \omega$$





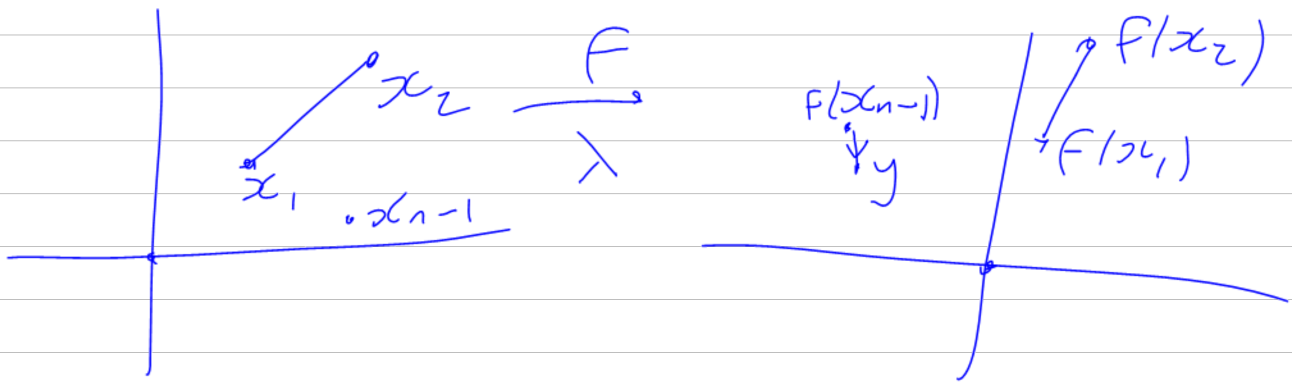


$$A \sim \pi \rho^2 = 2\pi r \epsilon$$

$$A \sim 2\pi r \cdot \epsilon$$

$$r^2 = \rho^2 + (r-\epsilon)^2$$

$$\rho = \sqrt{r^2-(r-\epsilon)^2} = \sqrt{2r\epsilon - \epsilon^2}$$
$$\sim \sqrt{2r\epsilon}$$




$$|\lambda(x_1 - x_2) - (f(x_1) - f(x_2))| < \epsilon |x_1 - x_2|$$

$x_0 = 0$ assume $x_0 \ldots x_{n-1}$ are def,

$$x_n = x_{n-1} + \lambda^{-1}(y - f(x_{n-1}))$$
$$x_{n+1} = x_n + \lambda^{-1}(y - f(x_n))$$

$$x_n - x_{n+1} = (x_{n-1} - x_n) - \lambda^{-1}(f(x_{n-1}) - f(x_n))$$
$$= \lambda^{-1}\underbrace{(\lambda(x_{n-1} - x_n) - (f(x_{n-1}) - f(x_n)))}_{\leq \epsilon |x_{n-1} - x_n|}$$

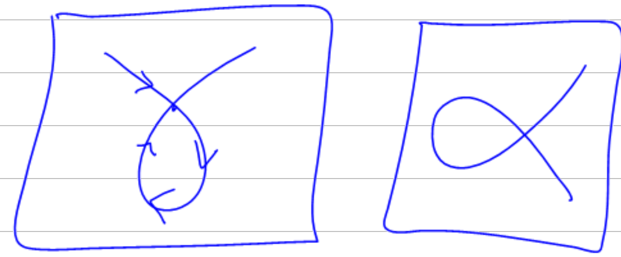
$\exists M$ s.t.

$$|x_n - x_{n+1}| \leq \overbrace{M\epsilon}^{<1} |x_{n-1} - x_n|$$

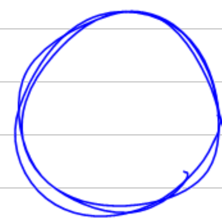
Q11 $\quad \gamma : \mathbb{R} \to \mathbb{R}^2 \quad$ $D\gamma$ is 1-1

$\gamma$ is not 1-1

$\gamma = \gamma$



$$\gamma(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



$$\gamma' = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

---

Q7

$$x + y + z = \sin(xyz)$$

$F(\bar{x}, \bar{y}) = 0$ $\quad \frac{\partial F}{\partial y}$ invertible near $x = y = z = 0$

$$F = x + y + z - \sin(xyz) = 0$$

$$\frac{\partial F}{\partial z} = 1 - xy\cos(xyz)\Big|_{x=y=z=0} = 1$$

$z = g(x, y)$ $\quad$ Find $\frac{\partial g}{\partial x}$ & $\frac{\partial g}{\partial y}$

$$F(x, y, g(x, y)) = 0$$

$$\left( \underbrace{\frac{\partial F}{\partial x}(x, y, g(x, y)) + \frac{\partial F}{\partial z}(\cdot)\frac{\partial g}{\partial x}}_{= 0} , \underbrace{\qquad\qquad}_{= 0} \right)$$

$$|\lambda(x_1-x_2)-(f(x_1)-f(x_2))| \leq \epsilon|x_1-x_2|$$

$$g(x)=f(\lambda^{-1}(x))$$
$$f(x)=g(\lambda(x))$$

$$\cancel{\text{Let } \begin{cases} g(x)=\lambda^{-1}(f(x)) \\ f(x)=\lambda(g(x))=y \end{cases}}$$

$$\cancel{|\lambda(x_1-x_2)-\lambda(g(x_1)-g(x_2))| \leq \epsilon|x_1-x_2|}$$

$$\cancel{|\lambda((x_1-x_2)-(g(x_1)-g(x_2)))| \leq \epsilon|x_1-x_2|}$$

$$\leq M|x_1-x_2-(g(x_1)-g(x_2))|$$

$$|\lambda(x_1-x_2)-(g(\lambda x_1)-g(\lambda x_2))| \leq \epsilon|x_1-x_2|$$

$$\lambda x_1 \to y_1 \qquad \lambda x_2 \to y_2$$

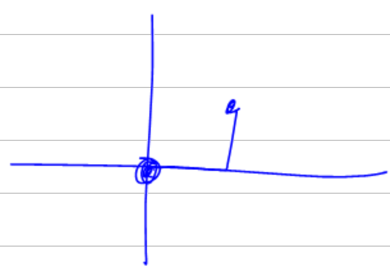
$$|(y_1-y_2)-(g(y_1)-g(y_2))| \leq \epsilon|\lambda^{-1}(y_1-y_2)|$$

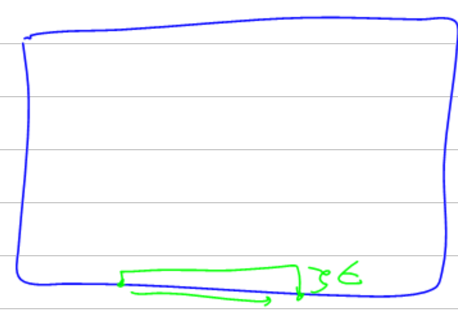
$$\leq M\epsilon|y_1-y_2|$$

$\Rightarrow g$ is invertible $\Rightarrow f$ is.

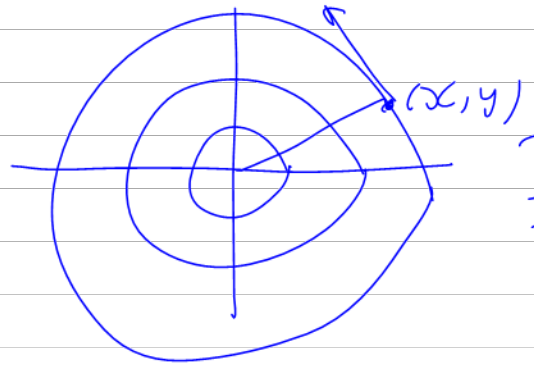
If $f'(a)=0$

$f$ is cont. diffable



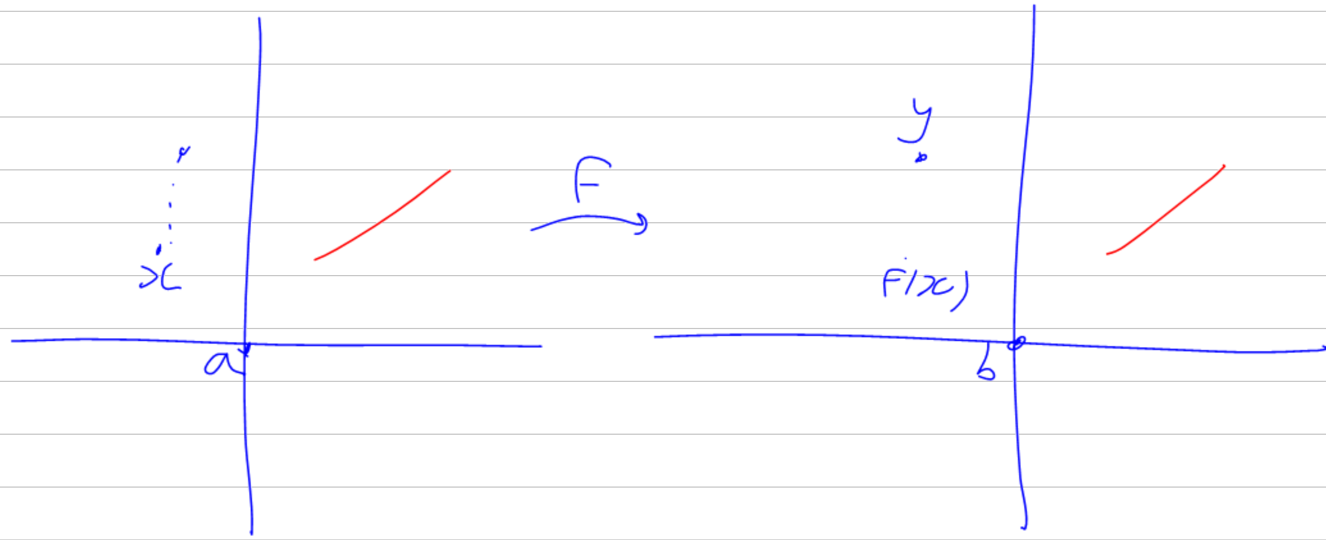


Q13




$$DF(x,y)(-y,x)$$

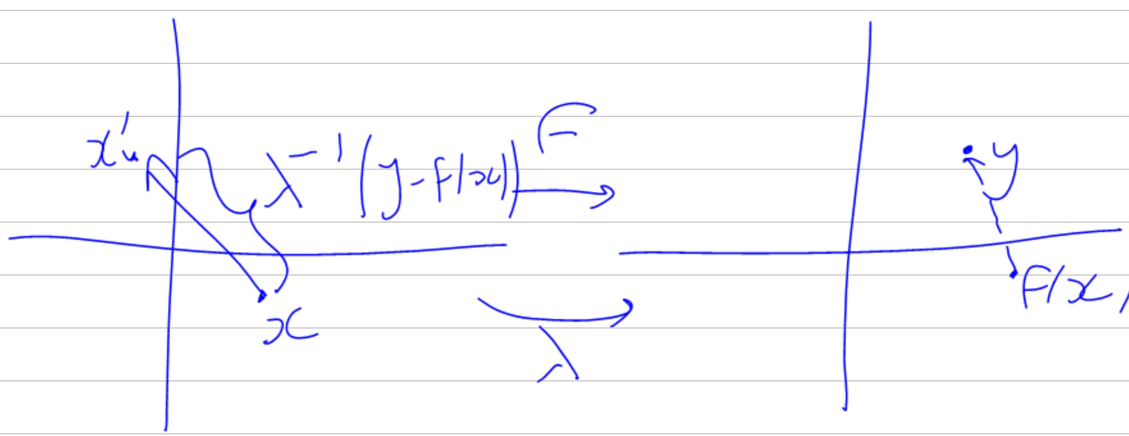
$$\frac{\partial}{\partial\theta}\left(F \circ T_\theta\right) = \cancel{x}\ 0$$




$$x' = x + y - f(x)$$

$$\left|(x_1 - x_2) - (F(x_1) - F(x_2))\right| < \epsilon \ldots$$

$$\Big|\,|x_1 - x_2| - |F(x_1) - F(x_2)|\,\Big| < \epsilon \ldots$$




$$|\lambda(x_1 - x_2) - (F(x_1) - F(x_2))| \leq \epsilon |x_1 - x_2|$$

$$x' = x + \lambda^{-1}(y - f(x))$$

$$x_0 = 257^{2020}$$

$$x'_n = x_{n-1} + \lambda^{-1}(y - f(x_{n-1}))$$

$$x_{n+1} = x_n + \lambda^{-1}(y - f(x_n))$$

$$x_n - x_{n+1} = (x_{n-1} - x_n) - \lambda^{-1}(f(x_{n-1}) - f(x_n))$$

$$= \lambda^{-1}\Big(\lambda(x_{n-1} - x_n) - (f(x_{n-1}) - f(x_n))\Big)$$

$$\Rightarrow \exists M \text{ s.t. } |\lambda^{-1} z| \le M|z| :$$

$$|x_n - x_{n+1}| \le M \in |x_{n-1} - x_n|$$

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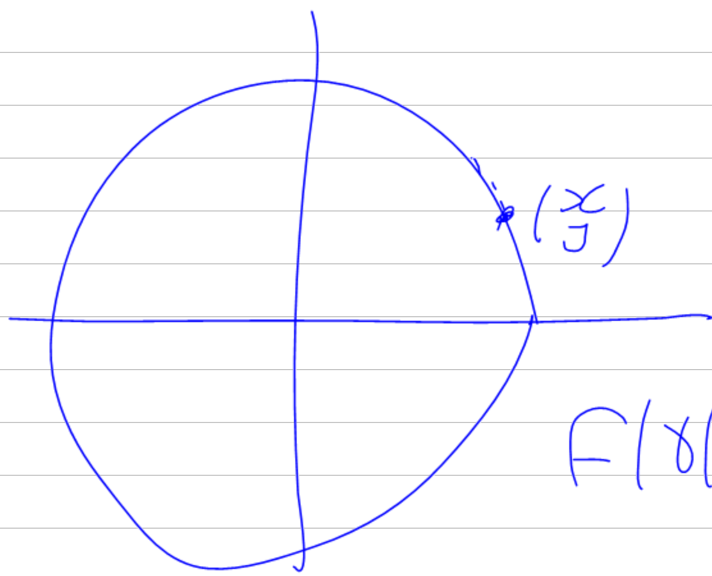



$$f \circ T_\theta = f$$

$$\gamma(\theta) = T_\theta\binom{x}{y}$$

$$f(\gamma(\theta)) = f\left(T_\theta\binom{x}{y}\right)$$

$$= f\binom{x}{y} \quad \text{indep of } \theta$$

$$0 = \frac{d}{d\theta} f(\gamma(\theta))\Big|_{\theta=0} = Df(x,y)\underline{\underline{\gamma'(0)}}$$

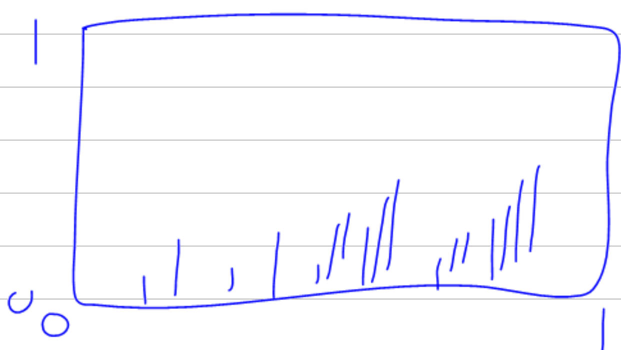
$$\gamma(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta\end{pmatrix}$$

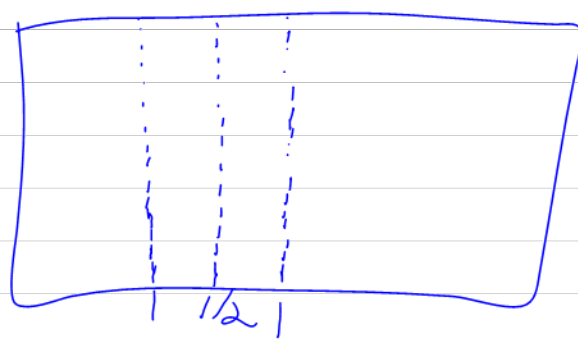
$$\int dx \,\overline{\underline{\int}} dy\, f(x,y) = \_ \quad \_ \quad \_$$

$\mathbb{Q}$

$$g(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

$$f(x,y) = 1 + g(x)\chi_{\mathbb{Q}}(y)$$




$$\left(\int f(x,y)\,dy\right)\chi_{\mathbb{Q}^c}(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

A cover A with N of balls of size $\epsilon$.

Suppose you wanted to cover A w/ balls of size $\frac{\epsilon}{n}$. How many do you need?

$$\text{Ans} \sim n^d N \qquad \log \text{Ans} = d \log n + \log N$$

$$\text{So } d = \lim_{n \to \infty} \frac{\log\left(\#\text{ of } \epsilon/n \text{ balls needed}\right)}{\log n}$$

$$\downarrow \lim_{k \to \infty} \frac{\log 2^k}{\log 3^k}$$

To cover C with balls of size $3^{-k}$, you need $2^k$ of them. $n = 3^k$

Hausdorff Dimension

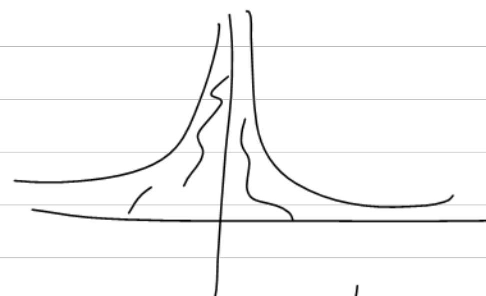
$A_i$ each is meas-0.

Given $\epsilon$, find $U_{ij}$ that cover $A_i$ & s.t. $\sum_{j=1}^{\infty} v(U_{ij}) < \frac{\epsilon}{2^i}$

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} v(U_{ij}) \le \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$$

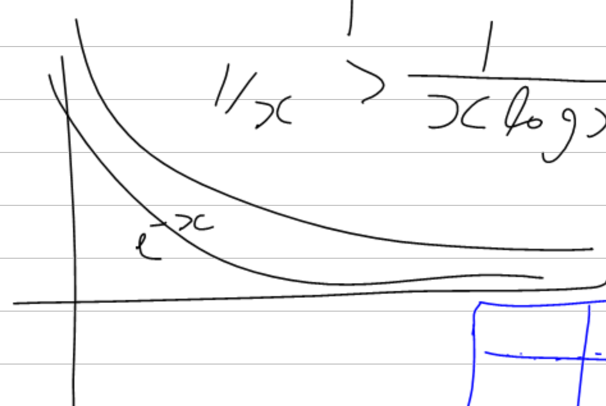
---

$\frac{1}{x^2}$ $\qquad$ $\frac{1}{\sqrt{|x|}}$



$\int_0^{\infty}$

$1/x > \frac{1}{x \log x} > \frac{1}{x^{1.01}}$





$\lambda \in \mathbb{Q}^c$

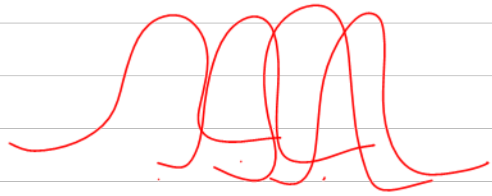
$$D = \{(a + \lambda b, b) : a, b \in \mathbb{Q}\}$$

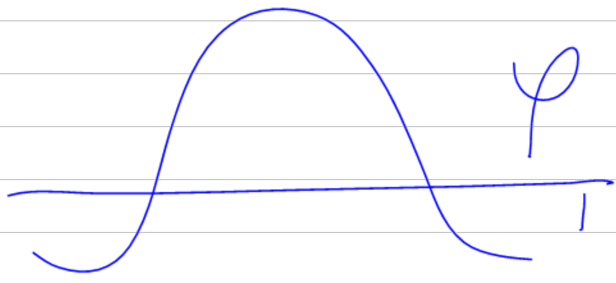
$F = 1_D$ $\qquad$ 1. If $y \in \mathbb{Q}$, $D \cap (\mathbb{R} \times \{y\})$ is dense

2. $\forall x \; |D \cap (\{x\} \times \mathbb{R})| \le 1$ $\qquad$ If $y \notin \mathbb{Q}$ $D \cap (\mathbb{R} \times \{y\}) = \emptyset$

$$a_1+\lambda b_1 = a_2+\lambda b_2 \Rightarrow \lambda = \frac{a_1-a_2}{b_2-b_1} \in \mathbb{Q}$$

$$\Rightarrow\Leftarrow$$

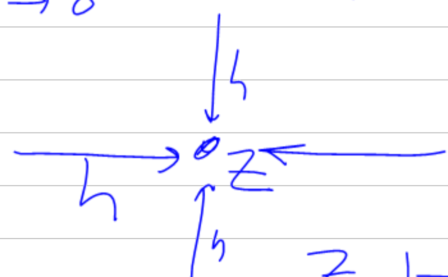





$\theta\circ\varphi$

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

$z, h \in \mathbb{C}$




$z \longmapsto \bar{z}$

$(x,y) \to (x,-y)$

$$\begin{array}{ccc} f: \mathbb{C} & \longrightarrow & \mathbb{C} \\ \| & & \| \\ \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \end{array}$$

$f(x) \longmapsto f(\phi(t))$

$\int f(x,y) \longmapsto f(\underline{\phi(t)}, \underline{\lambda(s)})$

$f(x,y) \longmapsto f(\underline{r}\cos\underline{\theta}, \underline{r}\sin\underline{\theta})$

$$\int_{M} d\omega = \int_{\partial M} \omega$$

$A = \mathbb{Q} \cap [0,1] \qquad \bar{A} = [0,1]$

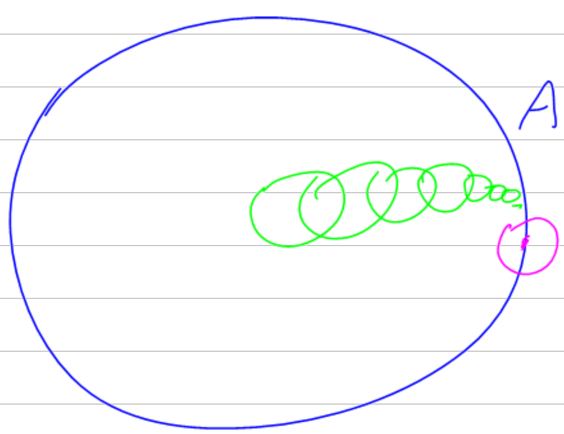
$\sum a_i$ is abs conv.

means $\sum |a_i|$ is conv.

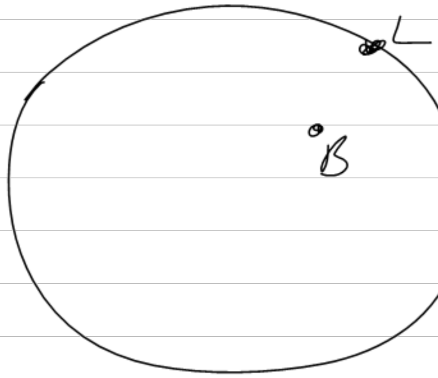
$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, -\frac{1}{7}, \frac{1}{8},$

$\left(-1 + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{4}\right) + \left(-\frac{1}{5} + \frac{1}{6}\right) + \ldots$

$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} - 1\right) + \left(\frac{1}{10} + \frac{1}{12} + \ldots - \frac{1}{3}\right) + \ldots$



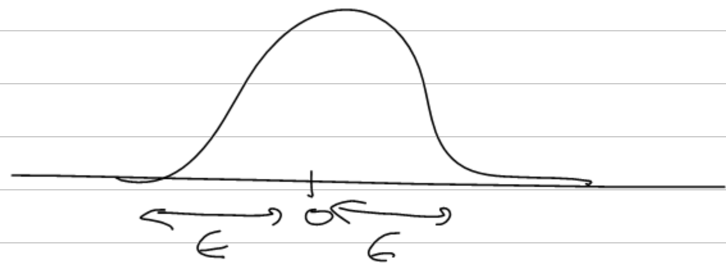
A is open




$V_L = 4V_B$

$V_L \qquad V_V$

$\beta_\epsilon$ :




$$\beta_{a,\epsilon} = \beta_{\epsilon^2}\left(|x-a|^2\right)$$

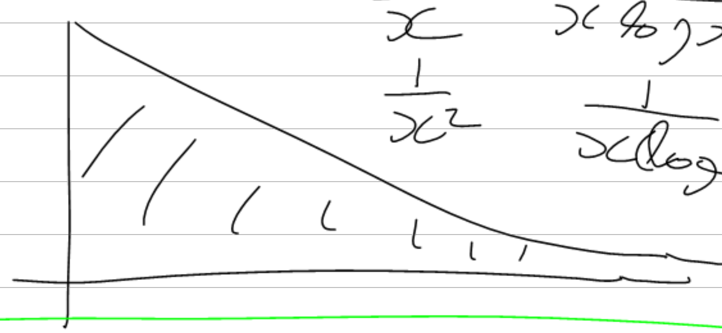
$$\int f<\infty \;\; \exists g \;\; \int g<\infty \;\; \frac{g}{f}\to\infty$$

$$\int f=\infty \;\; \exists g \;\; \int g=\infty \;\; \frac{g}{f}\to 0$$

$$\int_0^\infty f(x)\,dx$$

$$\frac{1}{x} \qquad \frac{1}{x\log x}$$

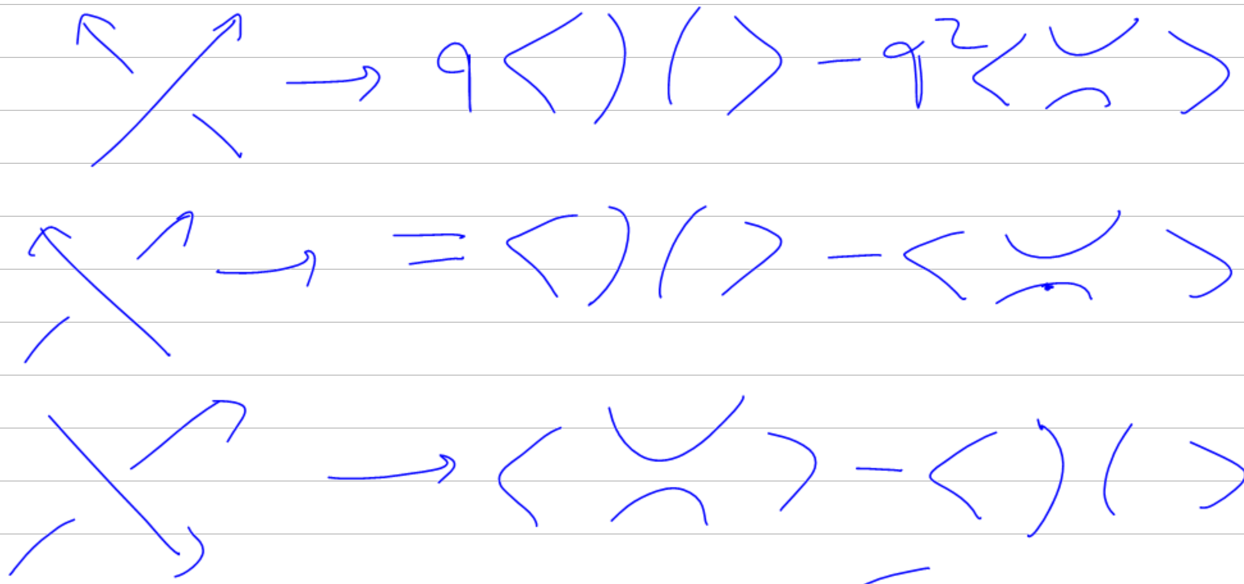
$$\frac{1}{x^2} \qquad \frac{1}{x(\log x)^2}$$

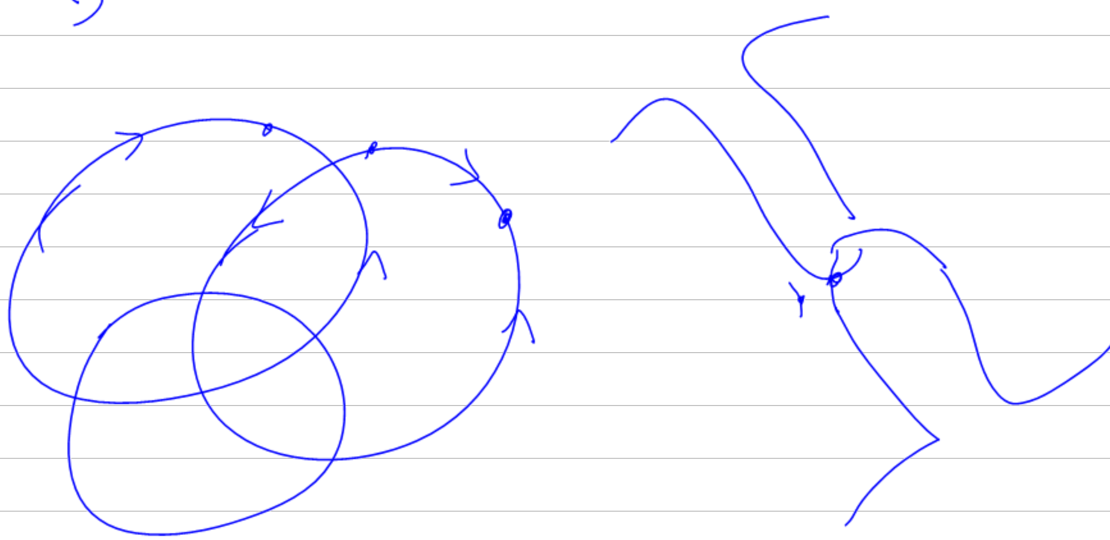


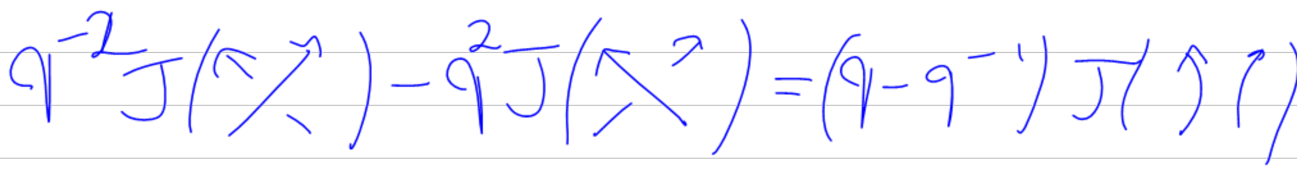
$$0 = h(x,y) = f(x,y,g(x,y))$$

$$\partial_x : \; 0 = D_1 f(x,y,g(x,y)) + D_3 f(-)\frac{\partial g}{\partial x}$$

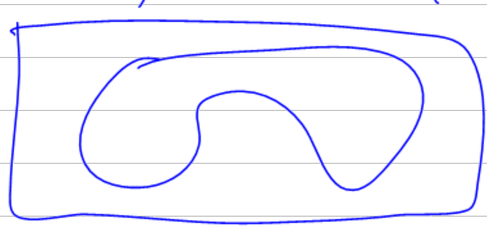
$$\frac{\partial g}{\partial x} = -\frac{D_1 f}{D_3 f}$$







$(1-2x)$ $\qquad\qquad$ $(1+2x)$



---

For every $x \in A$

$\exists\, U_x^{\text{open}}$ $x \in U_x$ $\exists\, g_x : U_x \to \mathbb{R}$ smooth

s.t. $g_x = f$ on $U_x$ Extend $g_x$ arbitrarily

$\mathcal{U} = \{V_x\}$ is an open cover beyond $U_x$
of $A$, so find a POI for $A$ subordinate to $\mathcal{U}$, $\Phi = \{\varphi_i\}$
Then for each $i$ there is some $x_i$ s.t. $\operatorname{supp} \varphi_i \subset U_{x_i}$

set $$\bar{F} = \sum_i \varphi_i \, g_{x_i}$$

loc-finiteness $\Rightarrow$ smooth.

$$\bar{F}(a) = \sum_i \varphi_i(a) g_{x_i}(a)$$
$a \in A$

$$= \sum_{i : a \in U_{x_i}} \varphi_i(a) \cancel{g_{x_i}(a)} \, f(a)$$

$$+ \cancel{\sum_{i : a \notin U_{x_i}} \varphi_i(a) g_{x_i}(a)}$$

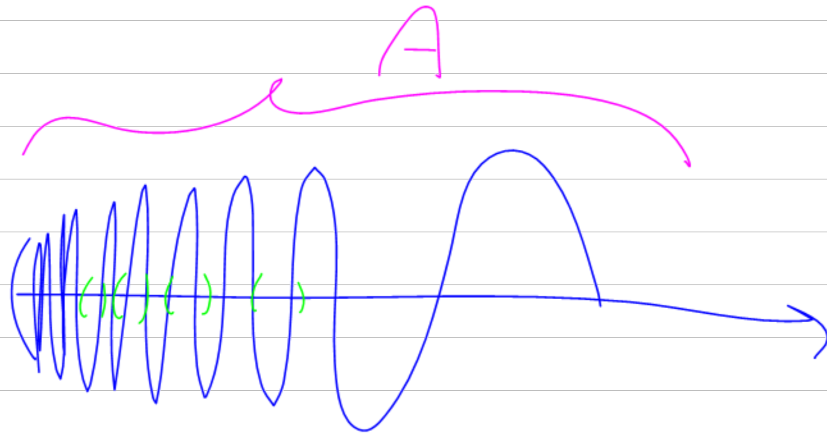
for all $a$
$\varphi_i(a) g_{x_i}(a)$
$= \varphi_i(a) f(a)$

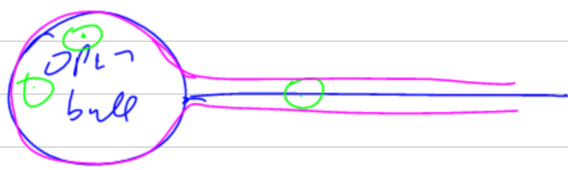
$$= \sum_{i : a \in U_{x_i}} \varphi_i(a) f(a)$$

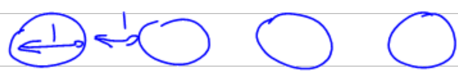
$$= f(a) \Big( \sum_{i : a \in U_{x_i}} \varphi_i(a) + \sum_{i : a \notin U_{x_i}} \underbrace{\varphi_i(a)}_{=0} \Big)$$

$$= f(x) \sum_i \varphi_i(x) = f(x)$$

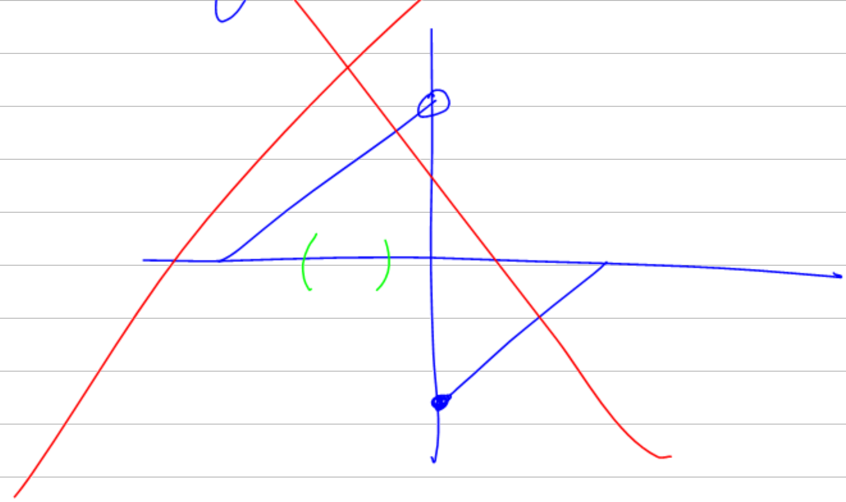
$\sin \frac{1}{x}$



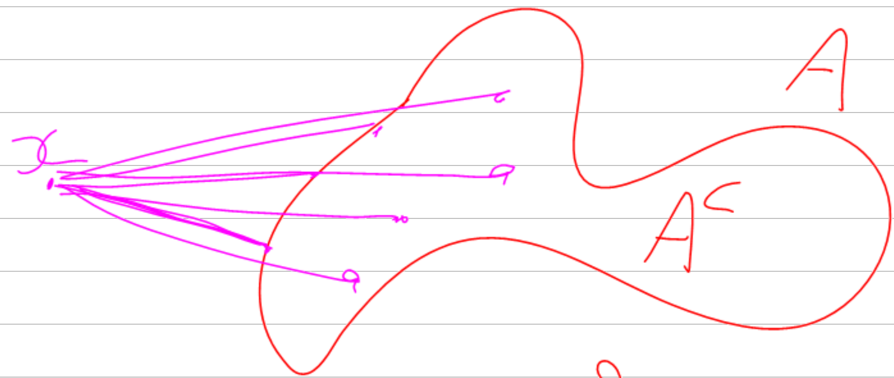



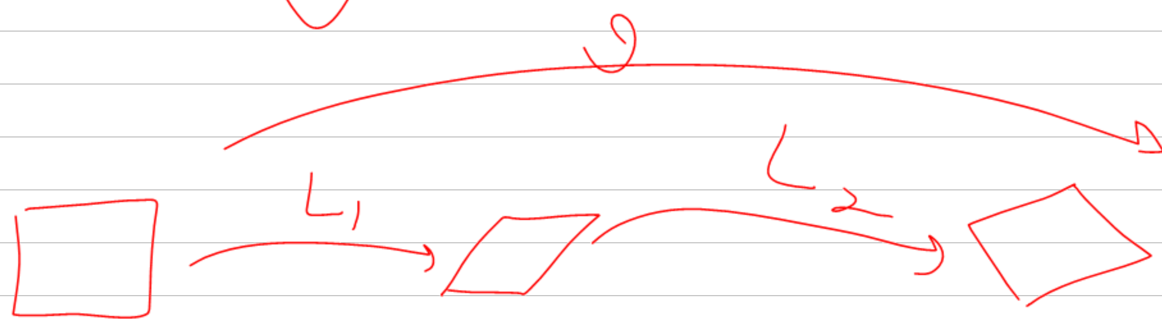
$$f(x) = \begin{cases} x - 1 & x \in [0, 1) \\ x + 1 & x \in (-1, 0) \end{cases}$$

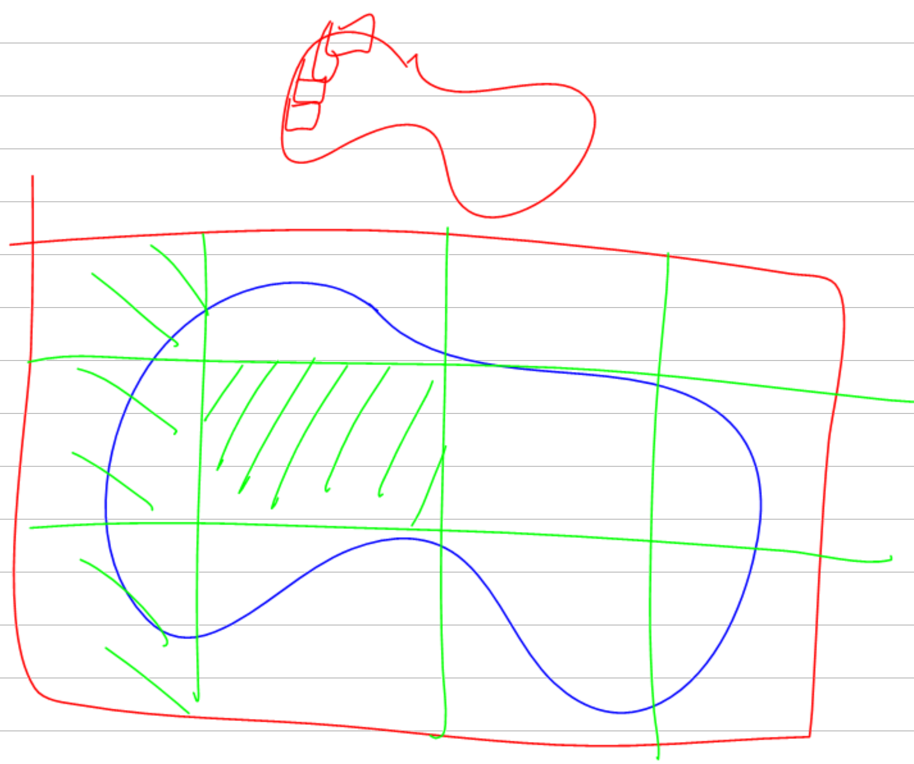


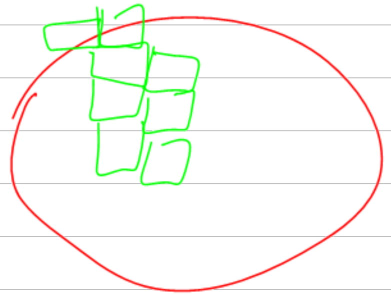
Winter Semester.

"Classification of simple groups"

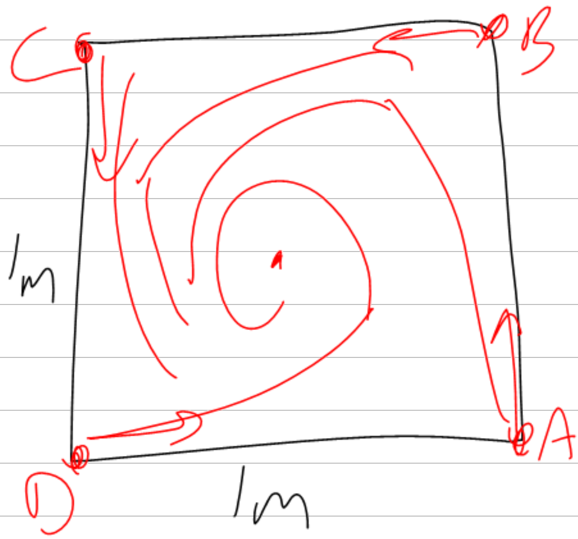


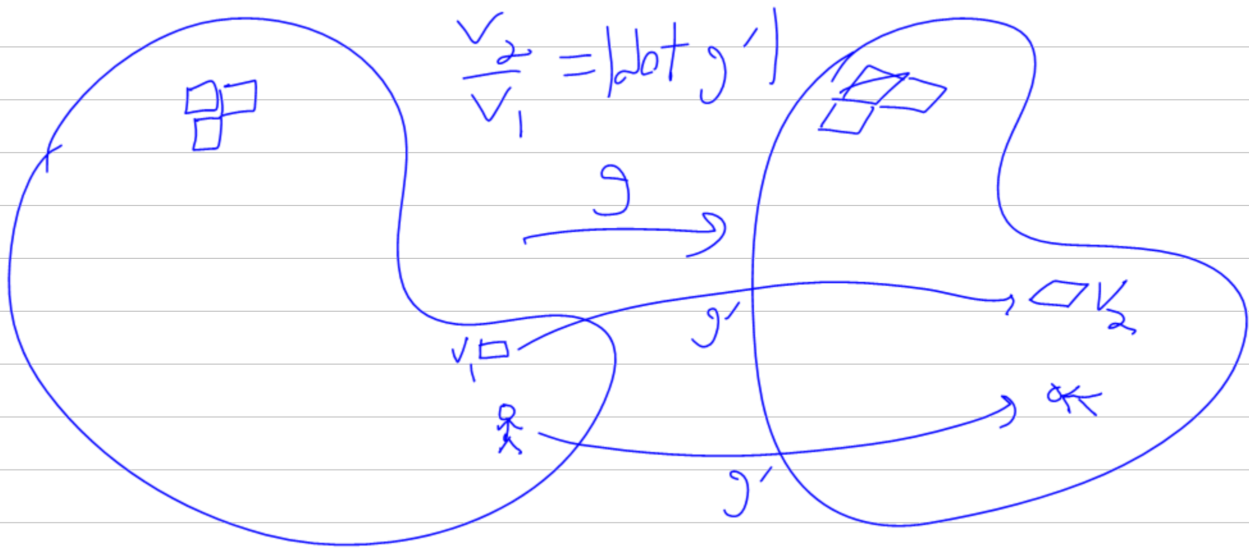



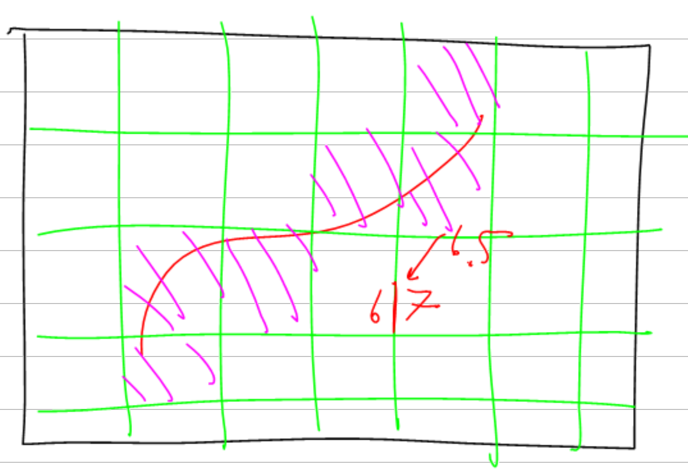


$$\frac{L}{\det(L)} = 7$$



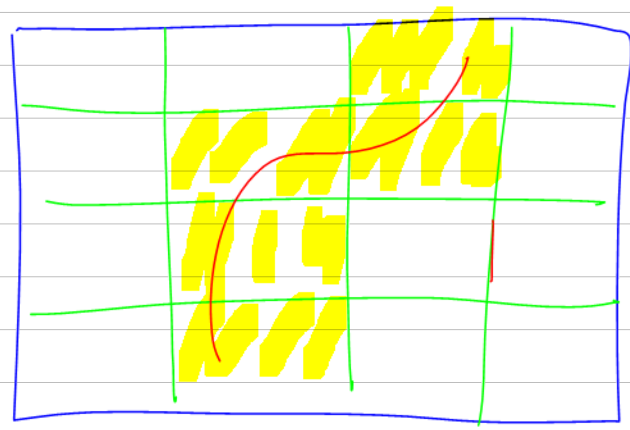

$V = 1\ m/s$



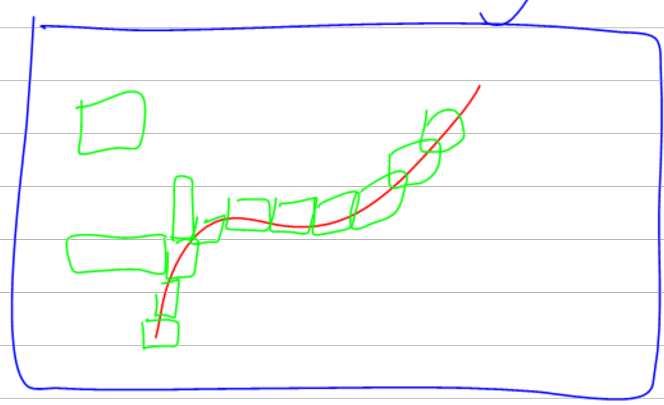

$$\int \psi_i f$$

$$A_n = \{x : O(f, x) \geq \tfrac{1}{n}\}$$



integ $\Rightarrow$ meas-0

meas-0 $\Rightarrow$ integ:



9. Prove that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is smooth and has an invertible differential at 0, then near 0 it can be written as a composition $T_n \circ g_n \circ \cdots \circ T_2 \circ g_2 \circ T_1 \circ g_1 \circ T_0$, where each $T_i$ is a "permutation map" that merely permutes the coordinates of $x = (x_1\ x_2\ \ldots\ x_n) \in \mathbb{R}^n$, and each $g_i$ changes the value of only the last coordinate; precisely, $g_i(x_1\ x_2\ \ldots\ x_n) = (x_1\ \ldots\ x_{n-1}\ h_i)$, where the $h_i\colon \mathbb{R}^n \to \mathbb{R}$ are smooth.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_k \end{pmatrix} \longrightarrow \begin{pmatrix} y_1 \\ \vdots \\ \cancel{y_k} \\ y_n \\ y_k \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_{k_1} \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ y_{k_2} \\ y_{k_1} \end{pmatrix} \longmapsto$$

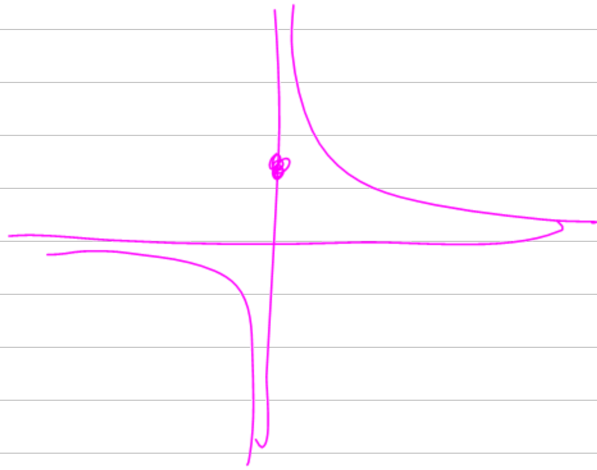
**Lemma**

Suppose $g: \mathbb{R}^n \to \mathbb{R}^n$, $g'(0)$ invertible

& $g\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ g_{k+1} \\ \vdots \\ g_n \end{pmatrix}$ then $\exists\, j > k$

st $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \overset{\alpha}{\longmapsto} \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ g_j \\ x_{k+1} \\ \vdots \\ x_n \end{pmatrix}$ is invertible.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \overset{\alpha_1}{\longmapsto} \begin{pmatrix} g_{j_1} \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \overset{\alpha_2}{\longmapsto} \begin{pmatrix} g_{j_1} \\ g_{j_2} \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow \begin{pmatrix} g_{j_1} \\ \text{and other } g\text{'s} \end{pmatrix}$$




$\varphi_i$ subordinate to a cover $\mathcal{U}$ of $A$

$$\#_1 = \sum \int \varphi_i |F| < \infty$$

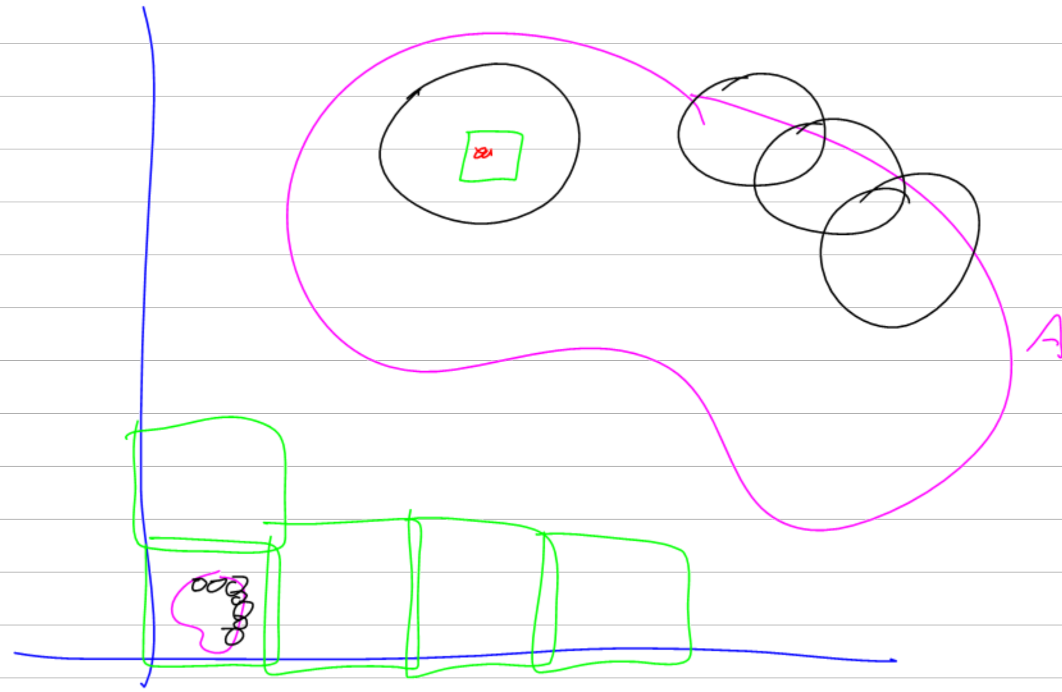
$\psi_i = \varphi_i \cdot \chi_B$ subordinate to $\mathcal{V} = \{U \cap B : U \in \mathcal{U}\}$

$$\sum \int \psi_i |F| = \sum \int \varphi_i \cdot \chi_B |F| \leq \#_1 < \infty$$

$\mathrm{disc}(F) \cap \mathrm{supp}\, \varphi_i$ is meas-0.

$\mathrm{disc}(F) \cap \left(\bigcup \mathrm{supp}\, \varphi_i\right)$ is meas-0

$\mathrm{disc}(F) \cap A$ is meas-0.





$$G_1 = \left\{(x,y) : \begin{array}{l} 0 \le x \le a \\ 0 \le y \le f(x) \end{array}\right\}$$

$$G_2 = \left\{(x,y) : \begin{array}{l} 0 \le y \le b \\ 0 \le x \le f^{-1}(y) \end{array}\right\}$$

then $G_1 \cup G_2 \supset R$

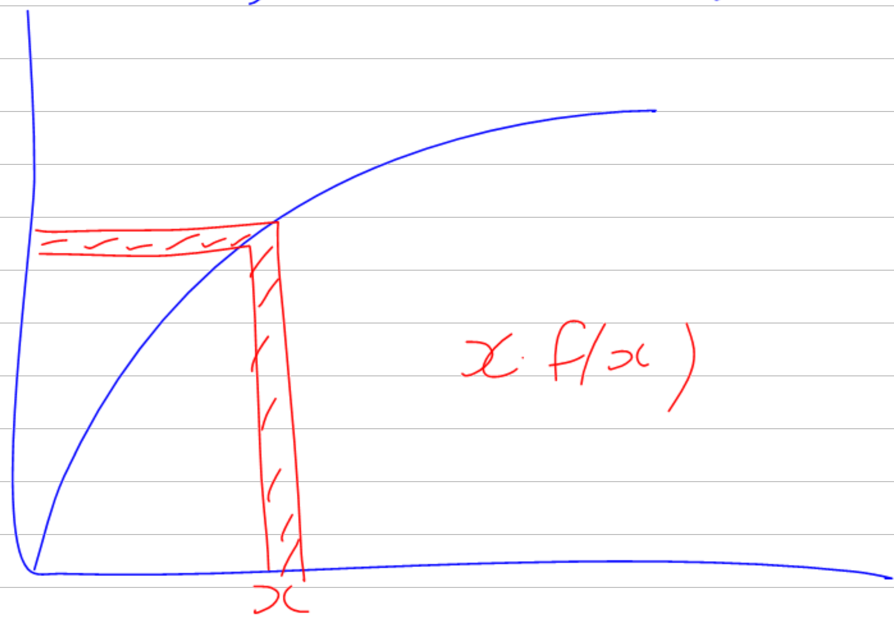
$$Vol(G_1) + Vol(G_2) \geq Vol(R) = ab$$

$$\int \chi_{G_1} = \int dx \underbrace{\int dy\, \chi_{G_1}(x,y)}_{f(x)} = \int dx\, f(x)$$

$$\int_0^b f^{-1}(y)\, dy = \int_0^c f^{-1} \circ f\, |\det f'|$$

$$= \int_0^c x f' \, dx =$$

$$\int (x f' + f) = \int (x f)'$$

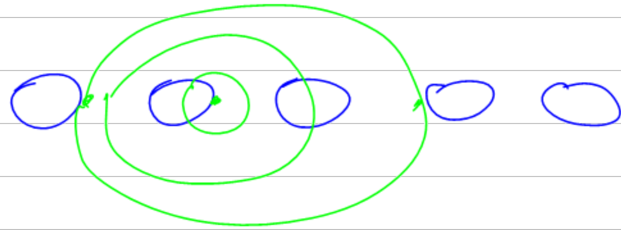


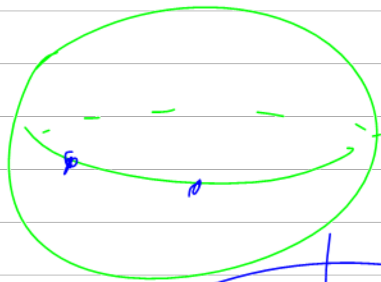

$$\int_U f = \sum_i \int_{\mathbb{R}} \varphi_i f$$

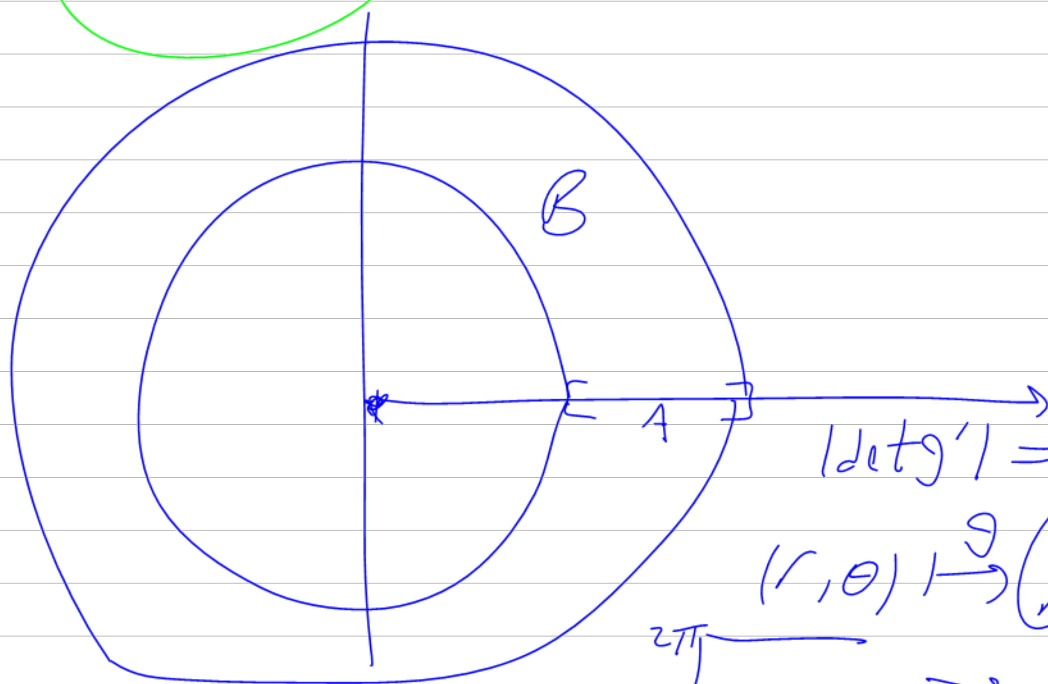
$$\mathbb{R} \subset \mathbb{R}^2$$

$$y = g(x)$$

$$\begin{tikzcd}
\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \arrow[r, "\alpha_1"] \arrow[rrrr, bend left, "g"] & \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_{n-1} \\ y_{j_1} \end{pmatrix} \arrow[r, "\alpha_2"] \arrow[rrr, bend left] & \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_{n-2} \\ y_{j_2} \\ y_{j_1} \end{pmatrix} \arrow[r] & \cdots & \begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{pmatrix}
\end{tikzcd}$$

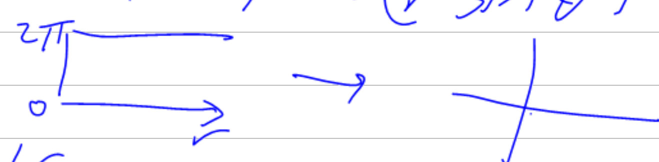







$|\det g'| = r$

$(r,\theta) \overset{g}{\longmapsto} \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

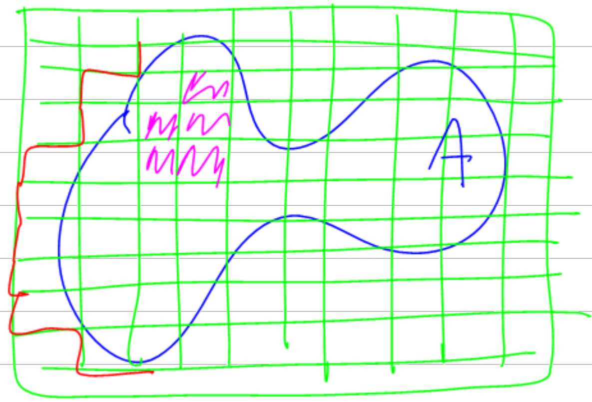



$$\mathrm{Vol}(B) = \int_{\mathbb{R}^2} \chi_B = \int \underbrace{\chi_B\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}}_{= \chi_A(r)} \cdot r \, dr \, d\theta$$

$$= \int r\,\chi_A(r)\,dr\,d\theta = 2\pi \int r\,\chi_A(r)\,dr$$

$$g(r,\theta,z) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$$

Q4




$\int \chi_A$ makes sense

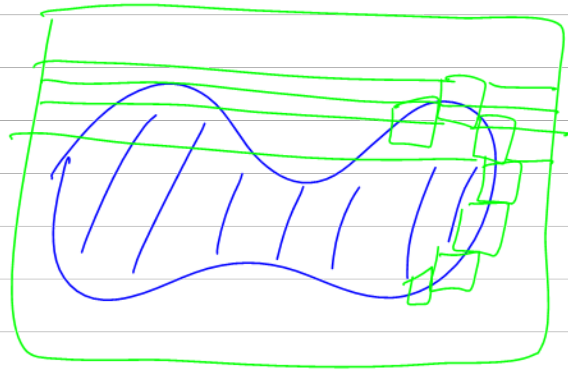
$\Updownarrow$

disc $\chi_A$ is meas-0

$\Updownarrow$

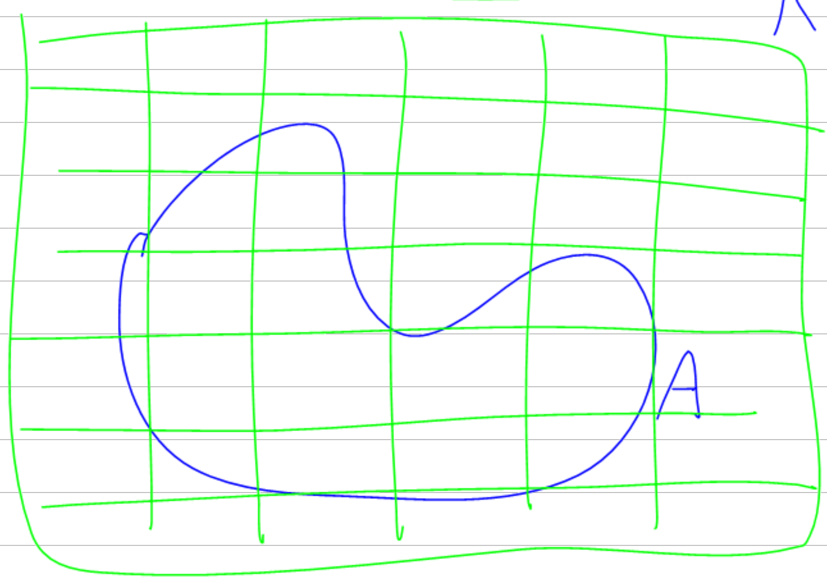
Bd $A$ is meas-0.

Bd $A$ is content-0



$\mathcal{R}$ = all rectangles in $P$ that touch $A$

$\mathcal{R}'$ = all rectangles that are contained in $A$.




$$U(\chi_A, P) - L(\chi_A, P) < \frac{1}{257}$$

$$\| $$

$$\sum_{\substack{S \in P \\ S \not\subset A \\ S \not\subset A^c}} \mathrm{Vol}(S) \cdot 1$$

---

$$V \oplus W = \left\{ (v, w) : \begin{matrix} v \in V \\ w \in W \end{matrix} \right\}$$

$$(0_V, 0_W)$$

$$O_{\sigma^*(V)} = \left( O_{\sigma^{-0}(V)}, O_{\sigma^{-1}(V)}, \dots \right)$$

$$T(v_i, v_j) = a_{ij}$$

$R_{abcd}$

$$\bigoplus_{k=0}^{\infty} T^k(V) = \left\{ \left( \underset{\Downarrow}{\overset{T^0}{}} T_0, \underset{\Downarrow}{\overset{T^1}{}} T_1, \underset{\Downarrow}{\overset{T^2}{}} T_2 \ldots \right) : \quad \right\}$$

$$(T_0, T_1 \ldots) \cdot (T_0', T_1', \ldots) = (\qquad\qquad)$$

$$V^* = \langle a, b, c, d, \ldots, z \rangle$$

$$T^k(V) = \langle \varphi_{i_1} \otimes \varphi_{i_2} \otimes \ldots \varphi_{i_k} \rangle$$

$$= \langle \underbrace{aabc}_{k}, \underbrace{dror}_{k=4}, \ldots$$

$$T^*(V) = \bigoplus T^k(V)$$

$= l.c.$ of words of all lengths.

$$= \left\{ 35ab + 2dror - \frac{1}{2}vid + \frac{22}{7}u \right\}$$

dror bar natan =

drorbarnatan

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "f/\!/g"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$\cancel{f \circ g}$ $\quad g \circ f$

$x /\!/ f \qquad \sqrt{}(a+b) = \sqrt{a+b}$

$(a+b)(c+d) = \overline{a+b}\;\overline{c+d}$

$\{ \quad \}$

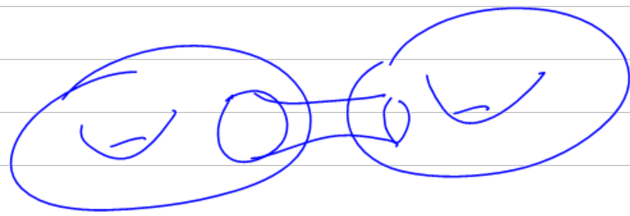
$((( \quad )) \quad ( \quad ))$

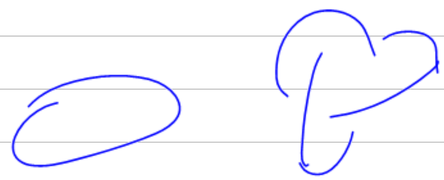
$\underline{v \in V} \qquad v = \sum a_i v_i$

$(a_i)$

$$\int f = \int f\,dx = \int f$$

$$\int \mathcal{D}q\, e^{-\frac{i}{\hbar}\int \frac{1}{2} m (\dot{q})^2 + V(q))}$$



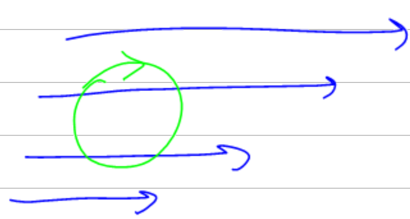


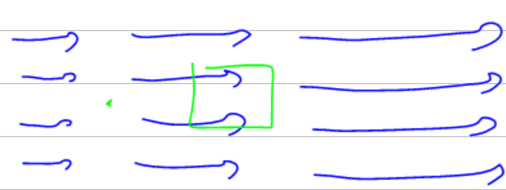
$\subset \mathbb{R}^3$

$F : \mathbb{R}^3 \to \mathbb{R}^3$

$$\int_{\partial\Sigma} F\cdot T\,ds = \int_{\Sigma} \underline{\mathrm{curl}(F)}\cdot\underline{\vec{n}}\,dA$$

$$(\nabla\times F)\cdot\vec{n}\,dA$$

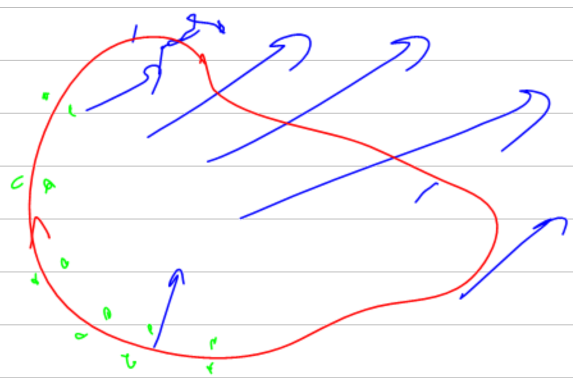


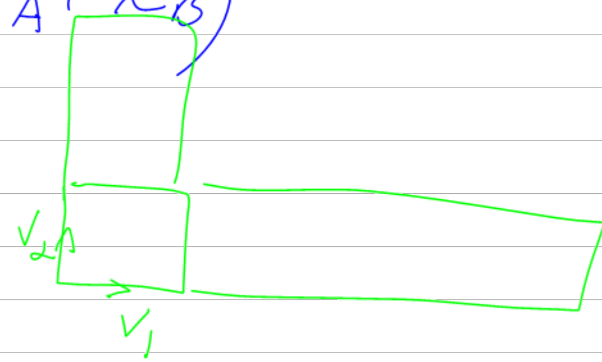


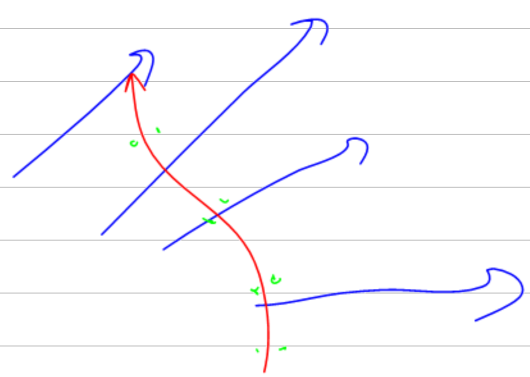
$A \cap B = \emptyset$


$$\int_{A\cup B} F = \int_A F + \int_B F$$

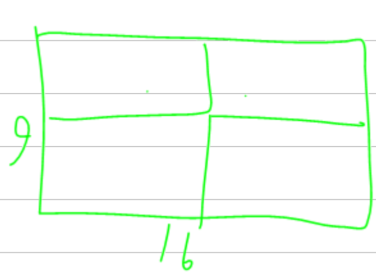
$$\int F \cdot \chi_{A\cup B} = \int F(\chi_A + \chi_B)$$





1. Check if it is possible to also record the whiteboard.




2. Try harder not to assign HW on material not yet covered.

3 change HW cycle to Mon → Mon.

$$\underbrace{(\lambda\wedge\eta)}_{}^{(u_1 \ldots u_{k+l})} = \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} (-1)^{\sigma}\,\lambda(u_{\sigma 1},\ldots u_{\sigma k})\cdot\eta(\qquad)$$

$k=2 \quad l=1$

$$\begin{array}{ll} 123 & +\lambda(u_1,u_2)\eta(u_3) \\ 132 & \\ \cancel{213} & -\lambda(u_2 u_1)\eta(u_3) \\ 231 & \\ \cancel{312} & \\ \cancel{321} & \end{array}$$

Aside $\quad \Delta^k_{nd} = \{1 \le i_1 \le i_2 \le i_3 \le \ldots \le i_k \le n\}$

$n=5 \quad k=7$

$$1223555 \longleftrightarrow 1\boxed{\overbrace{*2**3*45***}^{7+5-1}}$$

$$|\Delta^k_{nd}| = \binom{n+k-1}{k}$$

Chars in green box: $n+k-1$
of those $k$ are $*$'s.

$$W_I = \varphi_{i_1}\wedge\varphi_{i_2}\wedge\ldots\wedge\varphi_{i_k} \in \Lambda^k(V)$$

$$Alt\circ Alt = Alt$$

$$\mathrm{Alt}(T) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\sigma}\, \underbrace{T \circ \sigma^*}_{(-1)^{\sigma} T}$$

$$\omega_I = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$$

$$(\varphi_{i_1} + \varphi_{i_2}) \wedge \varphi_{i_2}$$

$$(\varphi_{i_1} + \varphi_{j})$$

---

$$(\omega_{12} + \omega_{34}) \wedge (\omega_{12} + \omega_{34})$$

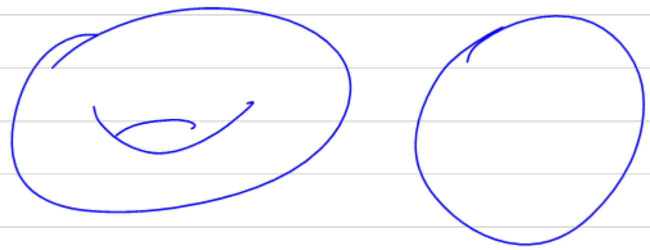
$$= 2\omega_{1234} \qquad \gamma = \omega_I$$

$F$ a v.f.

$D_F : \{\text{functions}\} \circlearrowleft$

$$D_F \circ D_G - D_G \circ D_F =$$

$$Fg = D_F g$$



$A \vee \neg A$

0.0000000

```
float x;
if (x=0) { _ _ _ }
    else{ }
```

$\sin(x)$

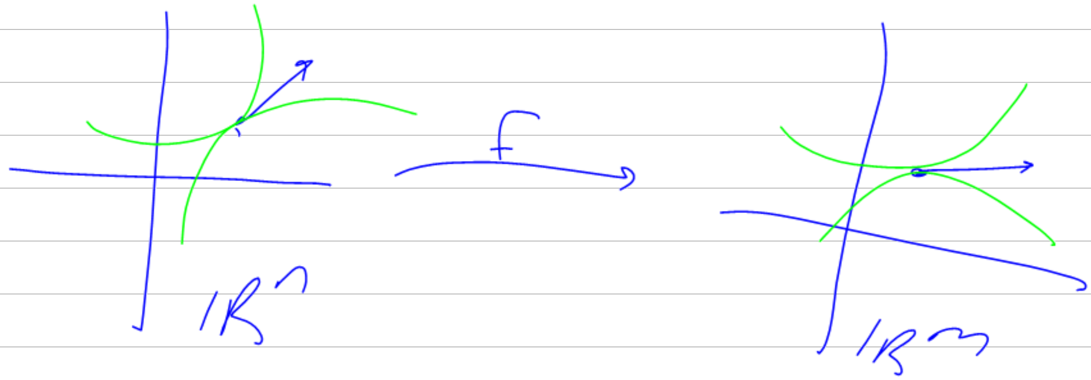
$x_n \in A_n$

$\mathcal{A} = \{ A \subset \mathbb{R} : A \neq \emptyset \}$

$f : \mathcal{A} \to \mathbb{R}$ st $f(A) \in A$

1.4142135 _ _ _

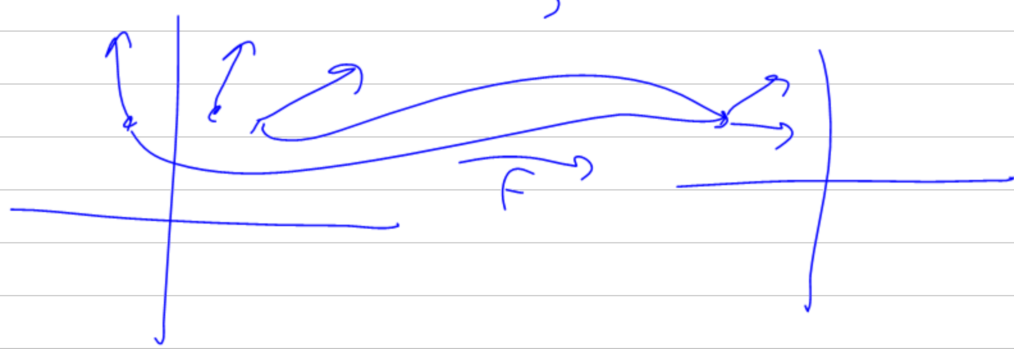
$f : \mathbb{R}^n \to \mathbb{R}^m$

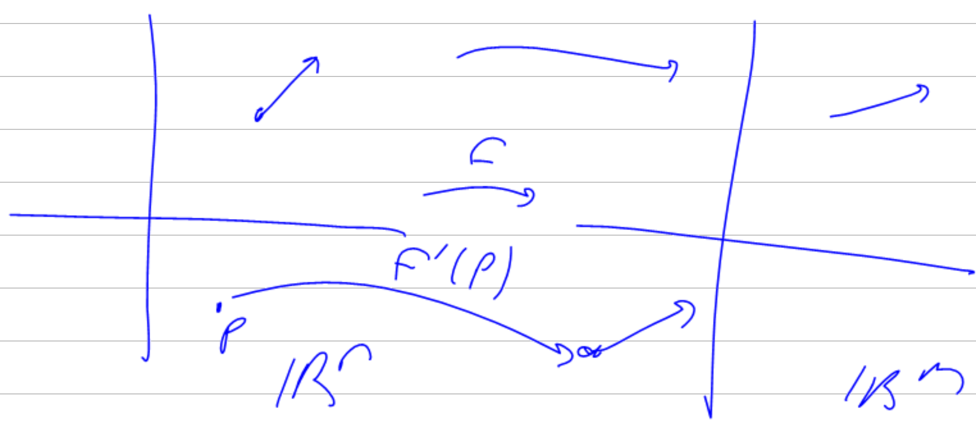


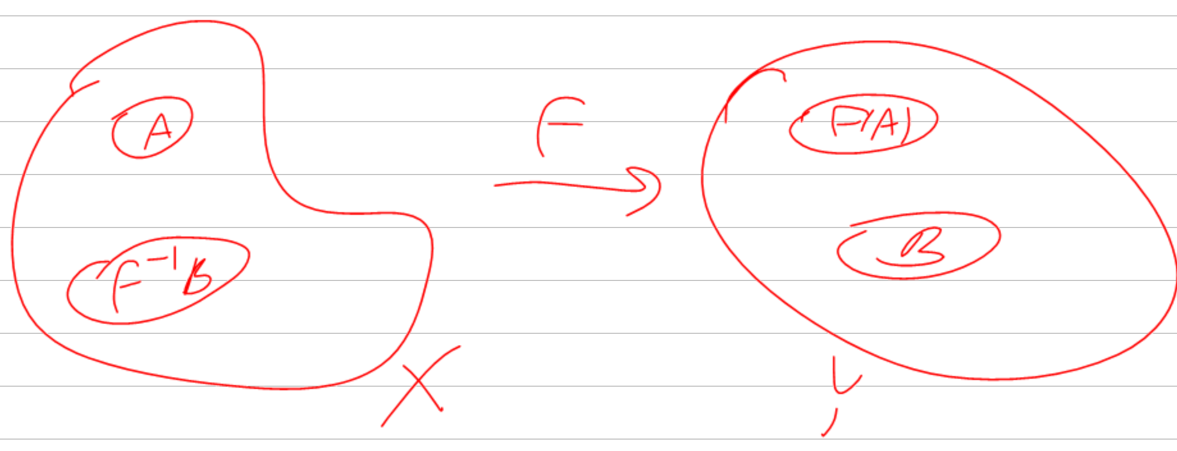

$$F_*(p,v) \neq (F(p), v)$$

$$F_*(p,v) = (F(p), F'(p)v)$$

$$F \neq F \qquad D_\xi F$$

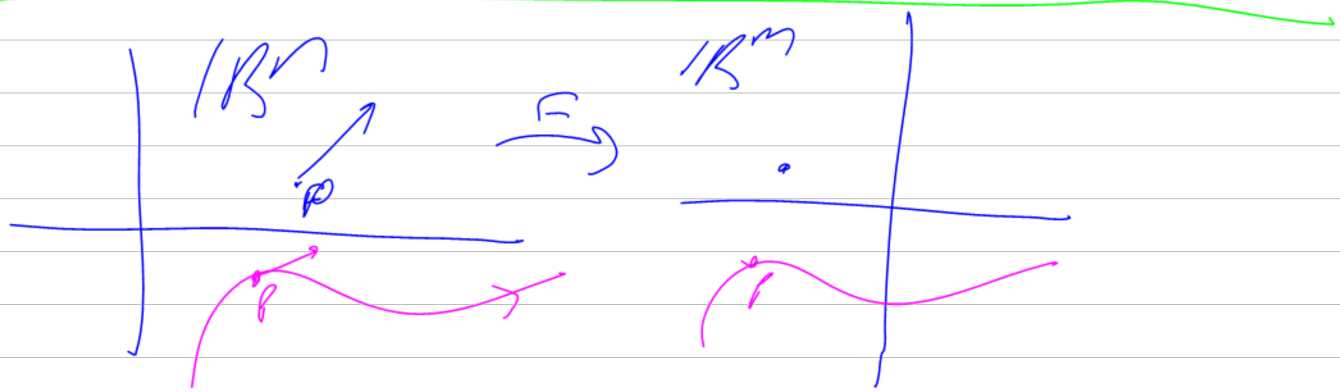


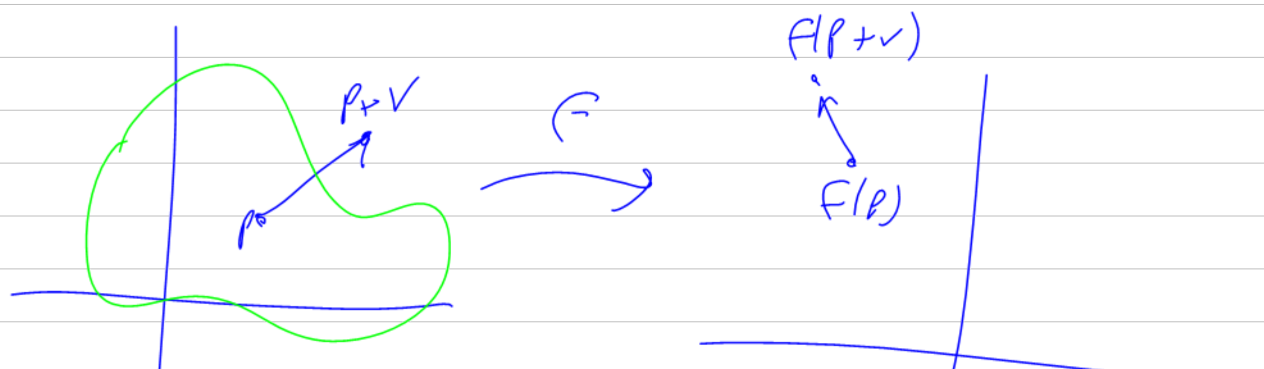



$$\int_0^{2\pi} d\theta \int_0^{\infty} dr \cdot r \cdot f(r)$$

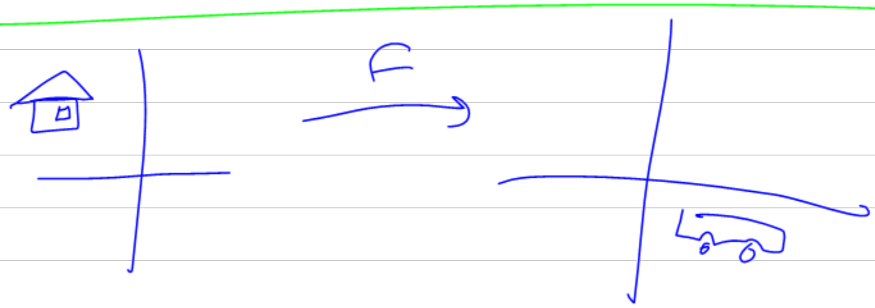
$$= 2\pi \int_0^{\infty} dr \; r \cdot f(r)$$

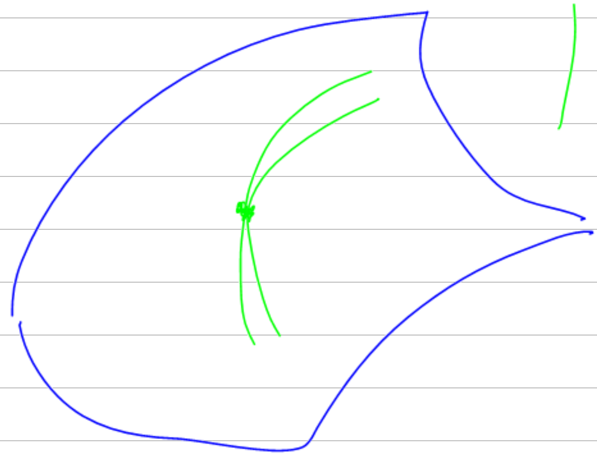



$$F_* \xi = (F(p), F(p+v) - F(p))$$

$$f(p+tv) = f(p) + t \underbrace{f'(p)\cdot v} + o(t)$$



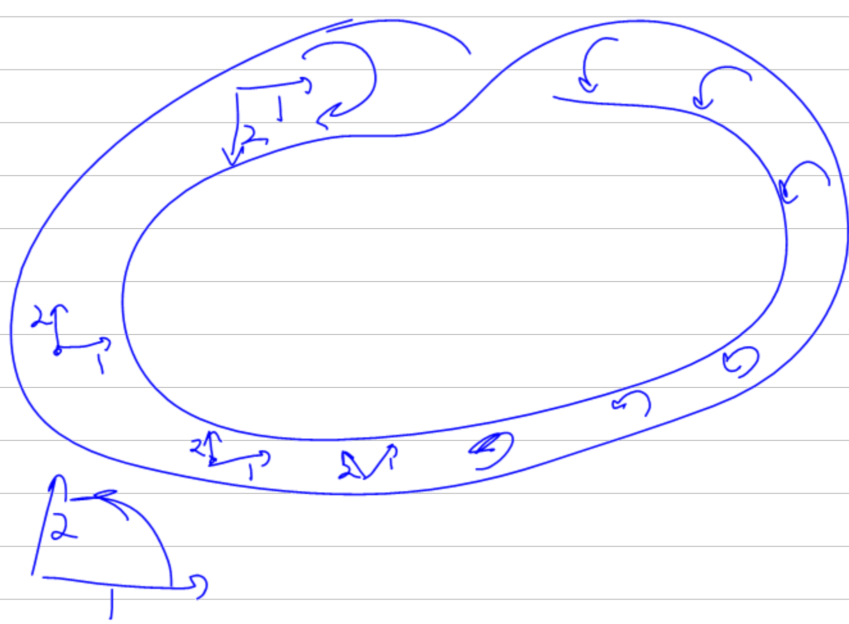


$$|\gamma_1(t)-\gamma_2(t)| \in o(t)$$

$$\sigma : \underline{n} \longrightarrow \underline{n}$$

$$(v_1 \ldots v_n) \longmapsto (v_{\sigma 1} \ldots v_{\sigma n})$$

$$\begin{tikzcd} \underline{n} \arrow[r, "\sigma"] \arrow[dr, "\sigma^* v = (v_{\sigma 1} \ldots v_{\sigma n})"'] & \underline{n} \arrow[d, "v = (v_1 \ldots v_n)"] \\ & V \end{tikzcd}$$



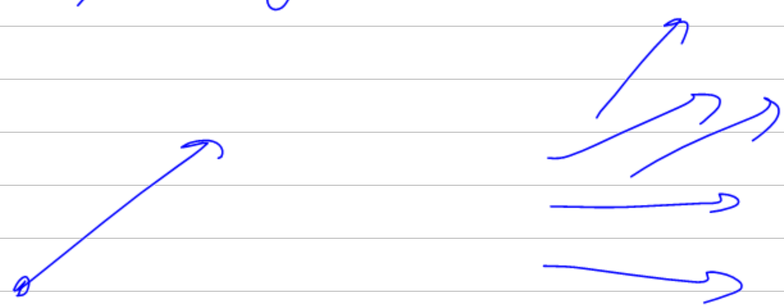
$\omega$ 2-tensor

$\eta$ 1-tensor

$$(\omega\cdot\eta)(x,y,z) = \omega(x,y)\,\eta(z)$$

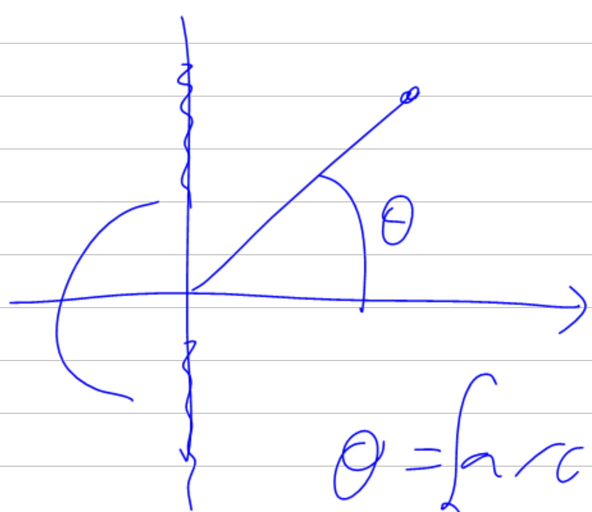
$$(\omega\cdot\eta)(x,z,y) = \omega(x,z)\cdot\eta(y)$$

$$(\omega\wedge\eta) = \sum_{\sigma} (-1)^{\sigma}\, \omega(\ldots)\,\eta(\ldots)$$



$$(\omega\wedge\eta)(p) = \omega(p)\wedge\eta(p)$$

---




$\theta\in(-\pi,\pi)$

$\arctan z \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$\theta = \begin{cases} \arctan\frac{y}{x} & x>0 \\ \pi/2 & x=0,\ y>0 \\ -\pi/2 & x=0\ \ y<0 \\ & y<0 \end{cases}$$

on $\mathbb{R}^3$

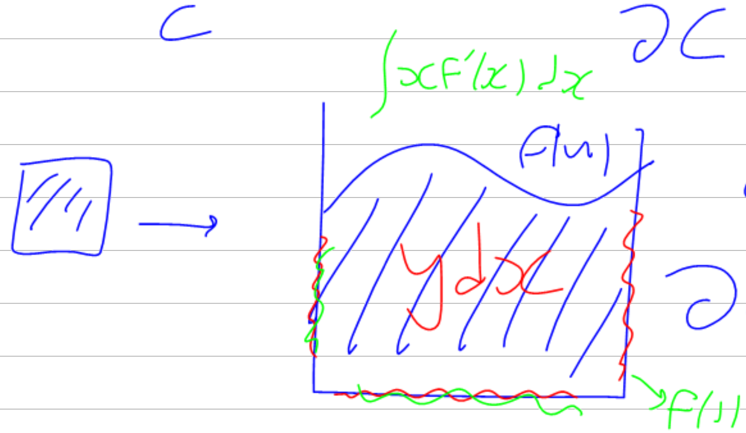
$$\begin{array}{ccccccc} \Lambda^0 & \xrightarrow[\text{grad}]{d} & \Lambda^1 & \xrightarrow[\text{curl}]{d} & \Lambda^2 & \xrightarrow[\text{div}]{d} & \Lambda^3 \\ \| & & \| & & \| & & \| \\ \text{Fnctns} & & \text{V.F.} & & \text{V.F.} & & \text{Fnctns} \\ & & & & & & g\,dx\wedge dy\wedge dz \\ 1 & & 3 & & 3 & & 1 \end{array}$$

---

$$\int_C dx\wedge dy = \int_{[0,1]^2} C^*(\underbrace{dx}_{u}\wedge \underbrace{dy}_{d(f(u)v)}) = \int_{[0,1]^2} f(u)\,du\wedge dv$$

$$d(f(u)v) = \underline{f'(u)v\,du} + f(u)\,dv$$

$$\int_C dx\wedge dy = \int_C d(-y\,dx) = -\int_{\partial C} y\,dx$$




$C:(u,v)\longmapsto(u,f(u)v)$

$\partial C = -C_{(1,0)} + C_{(1,1)} + C_{(2,0)} - C_{(2,1)}$

$C_{(1,0)}: I_t \longrightarrow \mathbb{R}^2$

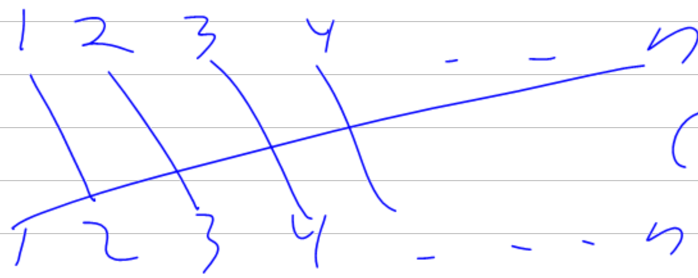
$t \longmapsto (0,t) \overset{C}{\longmapsto} (0,f(0)t) \xrightarrow{\int} 0$

$C_{(1,1)}: t \longmapsto (1,t) \longmapsto (1,f(1)t) \xrightarrow{\int} 0$

$$C_{(2,0)}: t \longmapsto (t,0) \longmapsto (t,0) \xrightarrow{\int} 0$$

$$C_{(2,1)}: t \longmapsto (t,1) \longmapsto (t,f(t)) \xrightarrow{\int} -\int f(t)\,dt$$

$$\int_{C_{(1,0)}} -y\,dx = -\int_{C_{(1,0)}} f(0)\cdot t\; d\cdot 0 = 0$$



$$(-1)^{n-1}$$

$$\begin{tikzcd}
\Omega^2(\mathbb{R}^3) \arrow[r, "d"] & \Omega^3(\mathbb{R}^3) \arrow[d, "\beta"] \\
V.F \arrow[u, "\alpha"] \arrow[r, "div"] & \text{Functions}
\end{tikzcd}$$

1. There is a bijection $\alpha : \left\{\begin{matrix}\text{smooth}\\ \text{V.F}\\ \text{on } \mathbb{R}^3\end{matrix}\right\} \to \Omega^2(\mathbb{R}^3)$ defined by

$$\alpha\begin{pmatrix}F_1\\F_2\\F_3\end{pmatrix} = F_1\,dx_2\wedge dx_3 + F_2\,dx_3\wedge dx_1 + F_3\,dx_1\wedge dx_2$$

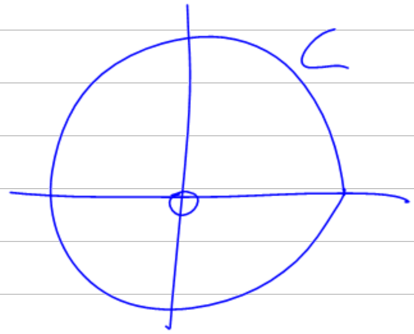
2. $\cdots$ $\beta : \Omega^3(\mathbb{R}^3) \to \left\{\begin{matrix}\text{smooth}\\ \text{Functions}\end{matrix}\right\}$

$$\beta : f\,dx_1\wedge dx_2\wedge dx_3 \longrightarrow f$$

<u>Claim</u> $\beta\circ d\circ\alpha = \text{div}$ <u>Pf</u> $\cdots$

$\omega$ exact, $\partial c = 0 \implies \int_c \omega = \int_c d\lambda = \int_{\partial c} \lambda = 0$

8d




$c: t \mapsto \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \end{pmatrix}$

$\int_c \omega = 2\pi \neq 0$

---

$$I^k \xrightarrow[\substack{\text{bij} \\ \det r' > 0}]{r} I^k \xrightarrow{c} W \subset \mathbb{R}^n$$

$$\int_{c \circ r} \omega \overset{?}{=} \int_c \omega = \int_{I^k} c^*\omega = \int_{I^k} f$$

$$\| $$

$$\int_{I^k} (c \circ r)^* \omega = \int_{I^k} r^*(c^*\omega) = \int_{I^k} r^*(f\,dx_1 \ldots dx_k)$$

$$c^*\omega = f\,dx_1 \ldots dx_k$$

$$= \int \underbrace{(f \circ r)(\det r')}\, dx_1 \ldots dx_k$$

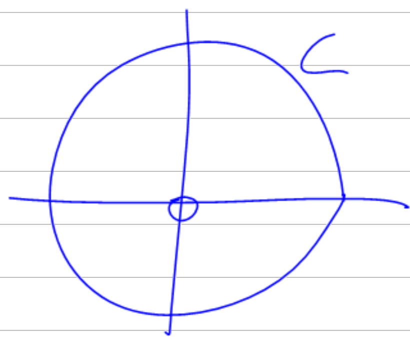
---

$L^*\phi = \phi A$

$$\begin{array}{ccc} \Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) \\ \updownarrow & & \updownarrow \\ \text{functions} & \xrightarrow{\text{grad}} & \text{vector fields} \end{array}$$

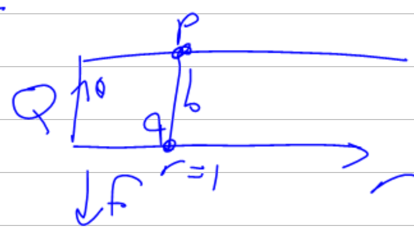
8d

$\omega$ exact, $\partial C = 0 \Rightarrow \int_C \omega = \int_C d\lambda = \int_{\partial C} \lambda = 0$

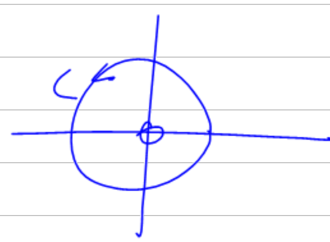


$C: t \mapsto \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \end{pmatrix}$

$\int_C \omega = 2\pi \neq 0$



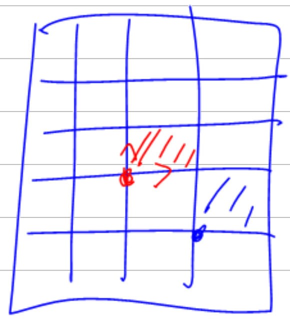
$$\int_C \omega = \int_{f_* b} \omega$$



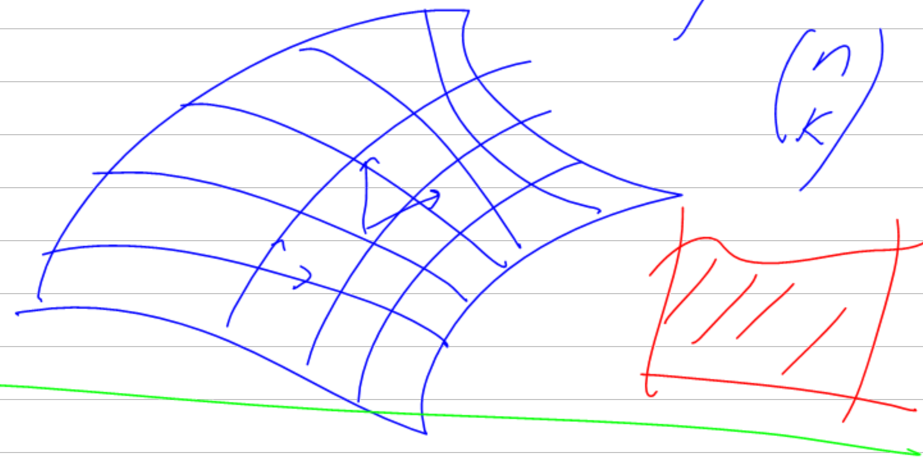
$$= \int_b f^* \omega = \int_b d\theta$$

$$= \int_{\partial b} \theta = \theta(p) - \theta(q) = 2\pi$$

$c^* w = f\, dx_1 \dots dx_k$ $\qquad w \in \Lambda^k(\mathbb{R}^n)$



$\binom{k}{k} = 1 \qquad \xrightarrow{\;c\;}$



$\binom{n}{k}$

---

$$\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{\;L\;} & \mathbb{R}^m \\
( \,) = v & \longmapsto & A v
\end{array}$$

$( \,) = w \longmapsto \phi \cdot w, \qquad \mathbb{R}^m \xrightarrow{\;\phi\;} \mathbb{R}$

$( \,) = v \longmapsto \underbrace{(\phi \cdot A)}_{(\text{———})} v, \qquad \mathbb{R}^n \xrightarrow{\;L^*\phi = \phi \circ L\;} \mathbb{R}$

---

$$d\omega(\xi_1, \xi_2) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \left( \int_{\partial \square} \omega \right)$$

---

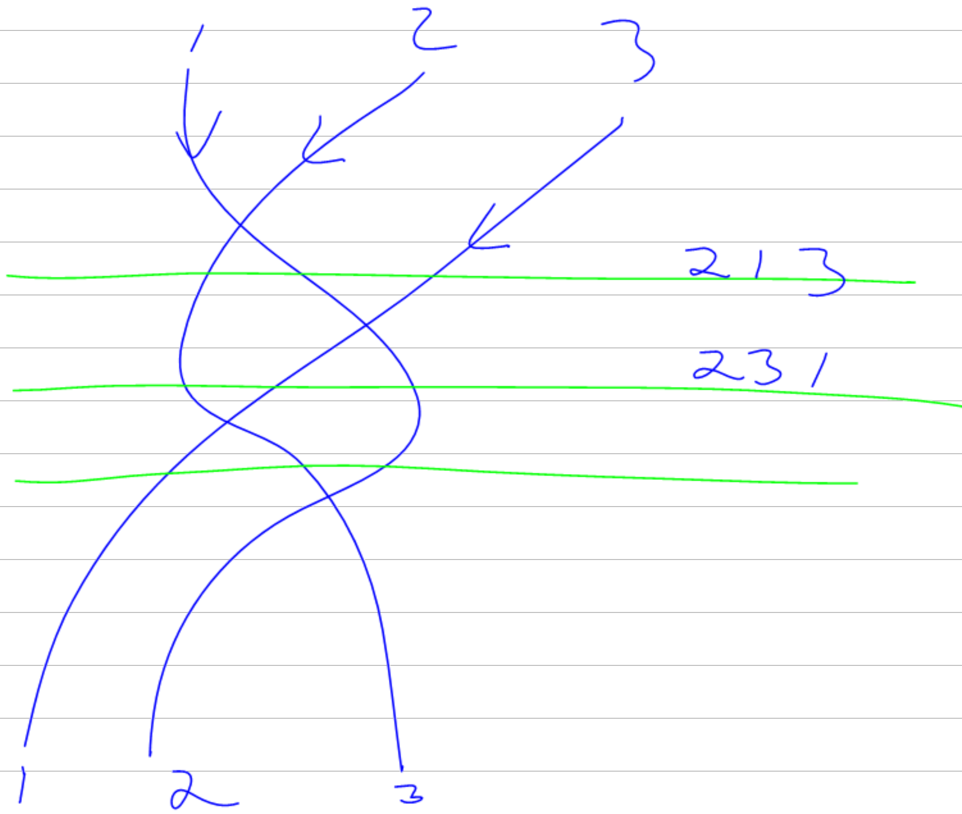
$\Lambda^k(V) : \quad (v_1, \dots, v_k) \longmapsto \mathbb{R}$

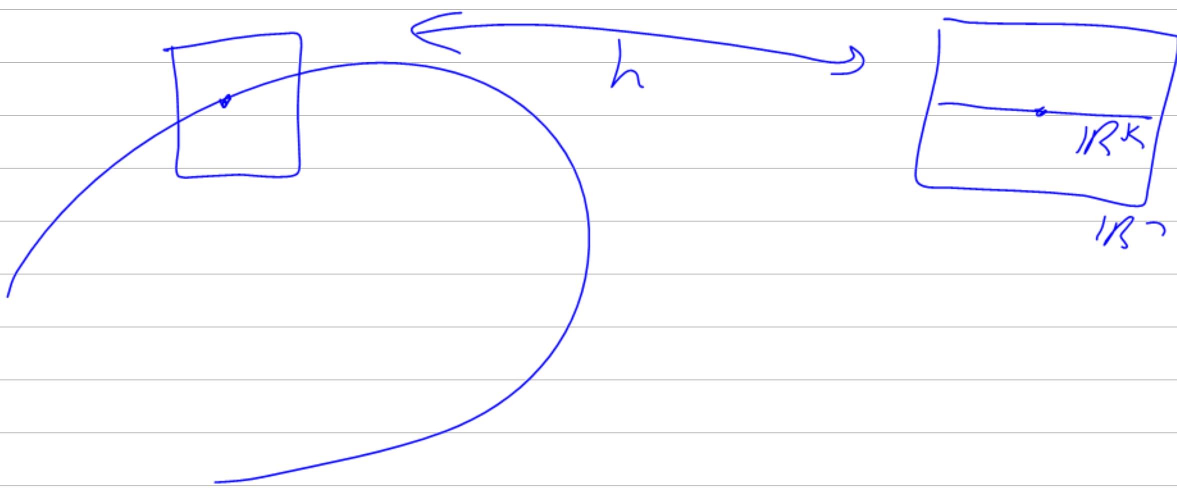
$k = 0 \qquad ( \,) \longmapsto \mathbb{R}$

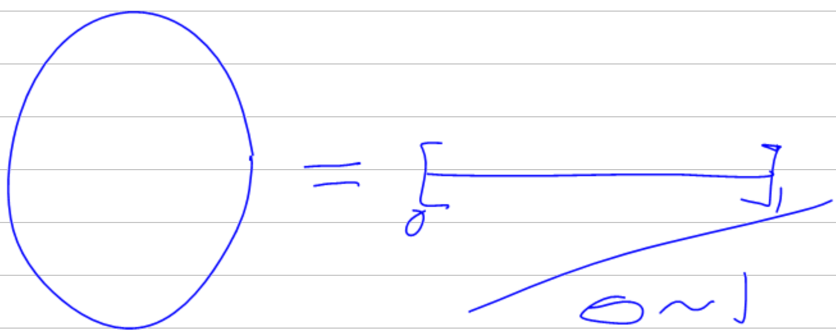
$\Lambda^{k=0}(V) \sim \mathbb{R}$

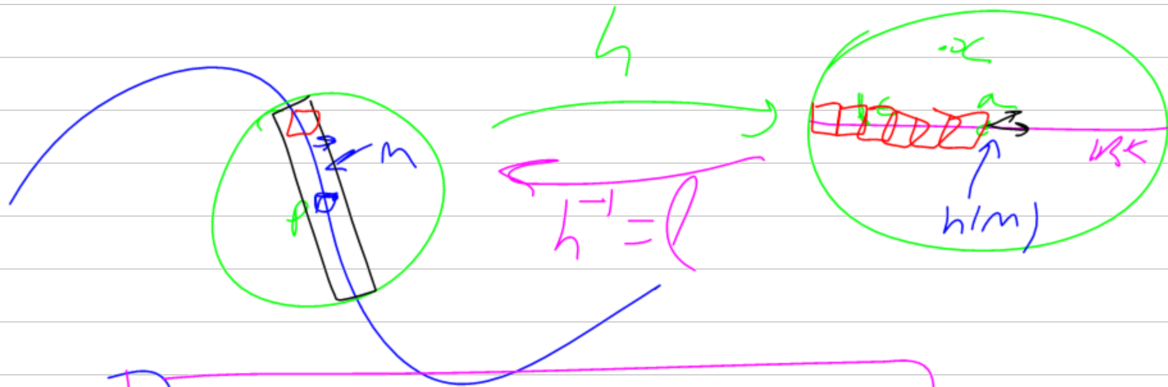
$\Lambda^k(\mathbb{R}^n)$ $\quad \mathbb{R}^n \ni p \longrightarrow \Lambda^k(T_p\mathbb{R}^n)$

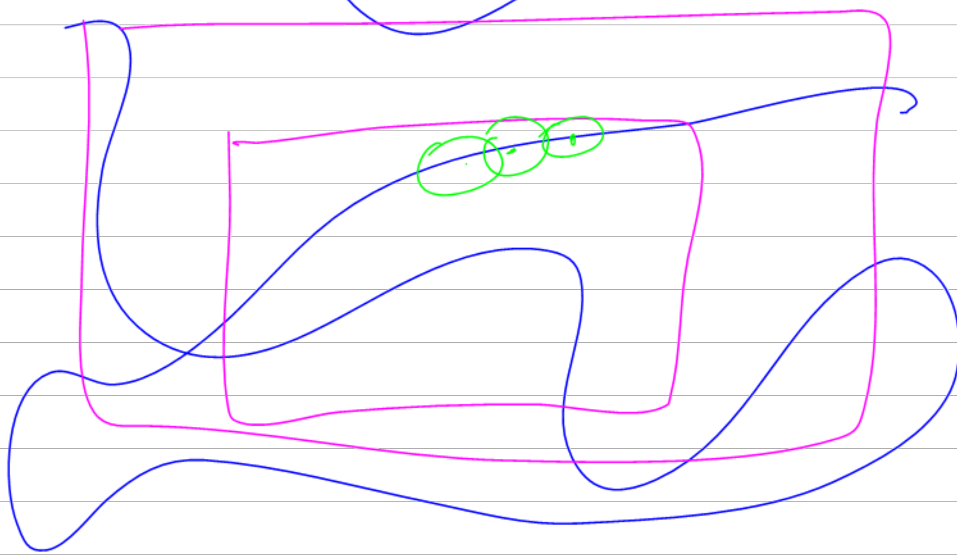
$k=0 \quad \mathbb{R}^n \ni p \to \Lambda^0(\_) \sim \mathbb{R}$

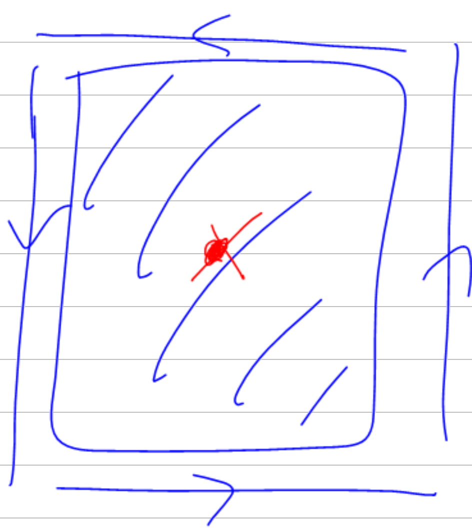


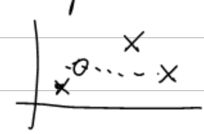





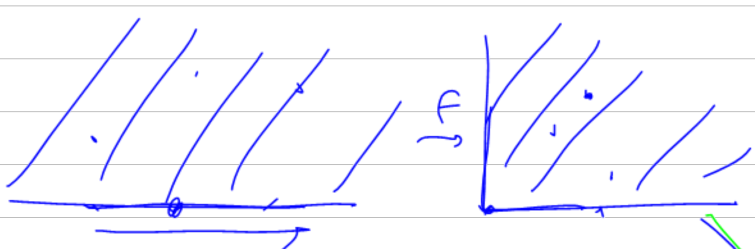





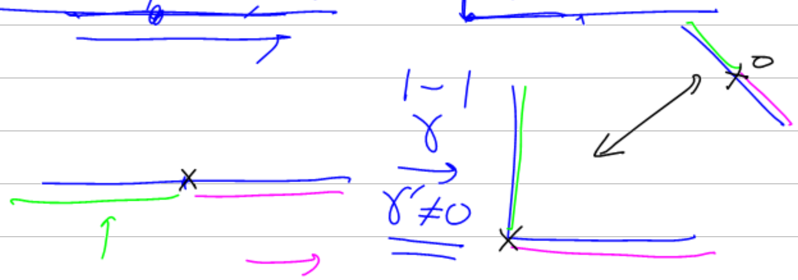
Claim A 1-1 cont
$\beta : \mathbb{R} \to \mathbb{R}$ is monotone



$f(\bar{A}) \subset \overline{f(A)}$

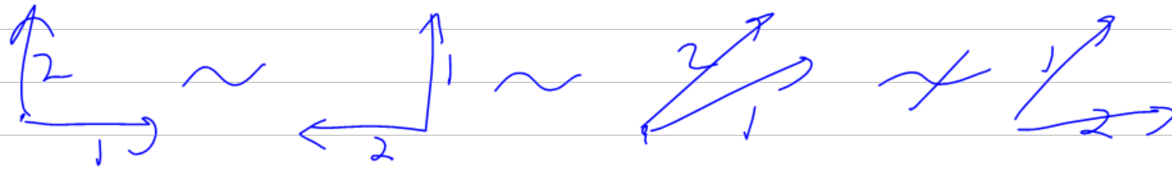


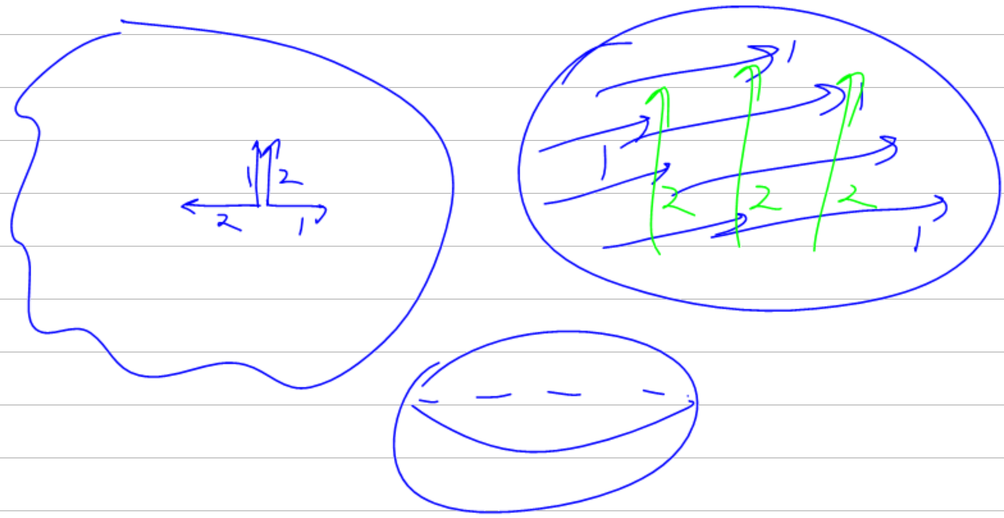


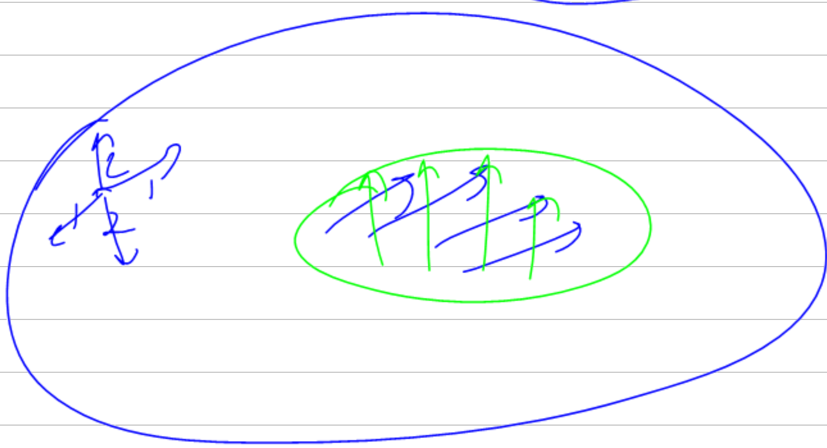


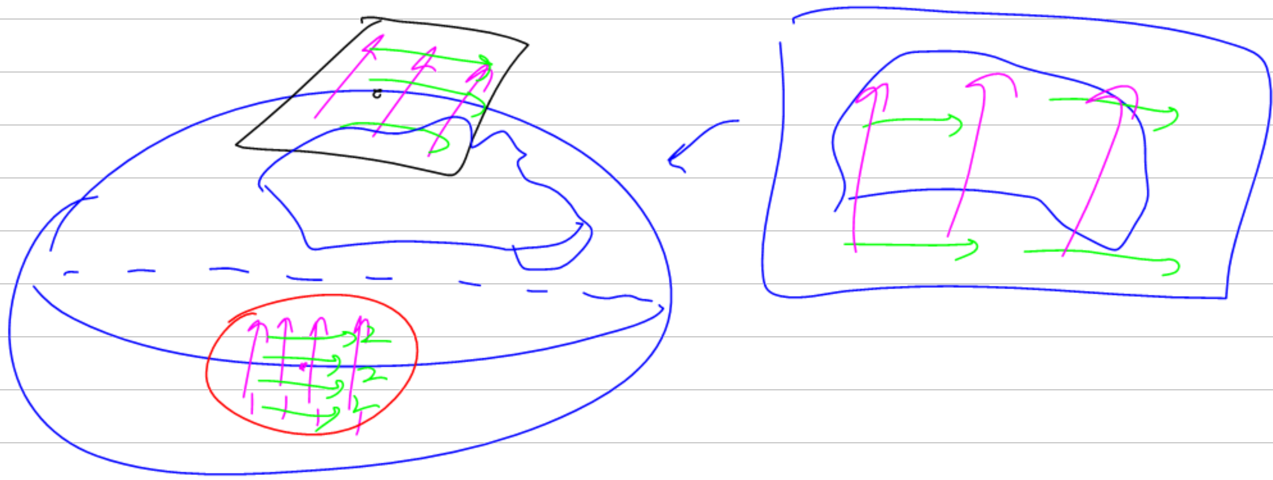

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h} = (*, 0)$$

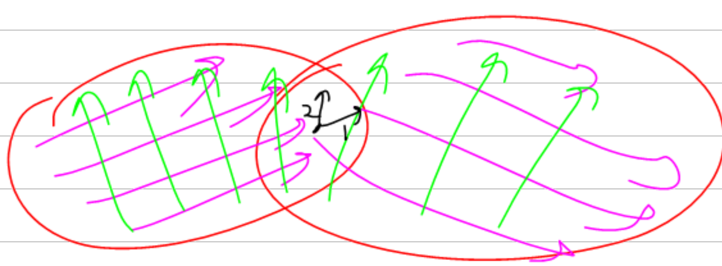
$(0, *)$

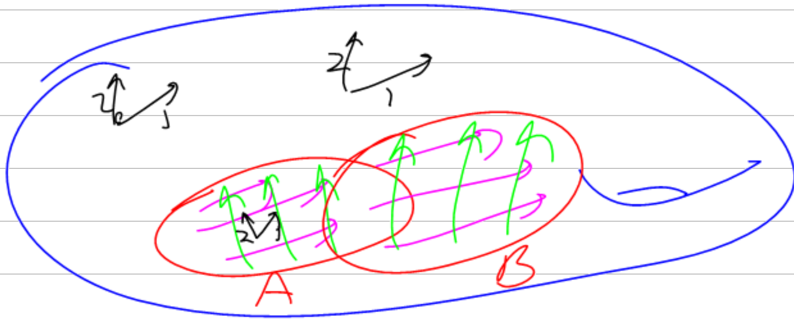


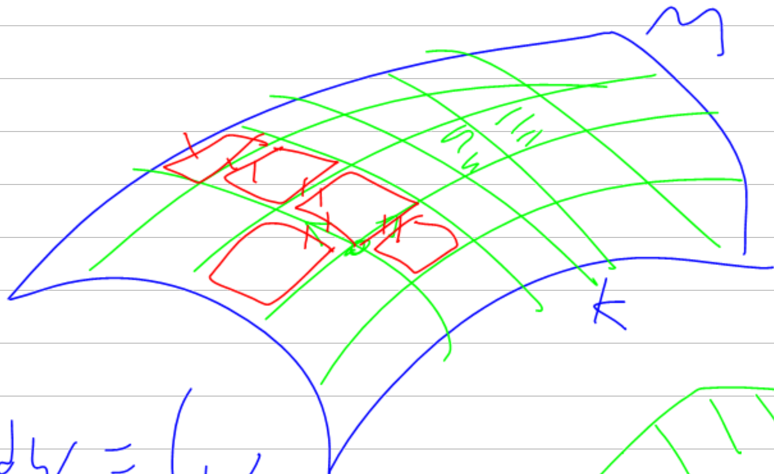






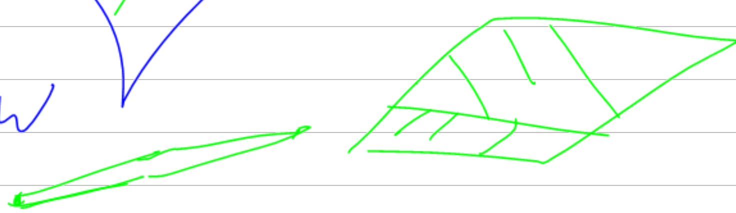




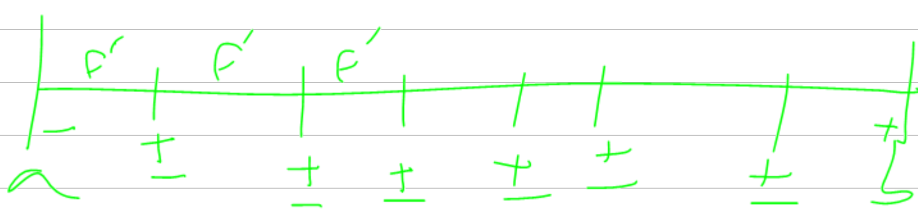



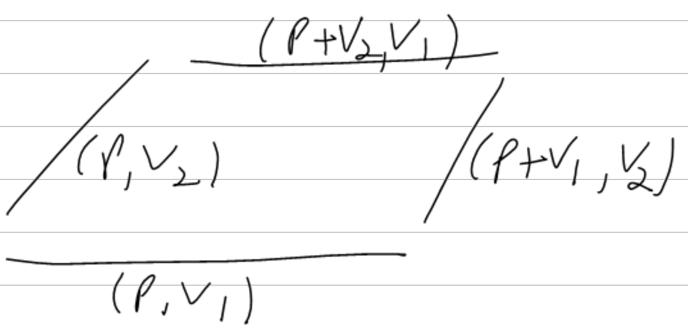

$\subset \mathbb{R}^n$

$$\int_M dw = \int_{\partial M} w$$



$$\int_{[a,b]} F' = \int_{\partial [a,b]} F = F\Big|_a^b$$




$\xi_i = (P, V_i)$

$$\int_Q dw = \int_{\partial Q} w$$

$$dw(\xi_1, \xi_2) \sim \begin{matrix} \pm w_p(v_1) \\ \pm w_p(v_2) \\ \pm w_{p+v_1}(v_2) \\ \pm w_{p+v_2}(v_1) \end{matrix}$$

---

$$\begin{matrix} \Lambda^1(\mathbb{R}^3) & \xrightarrow{d} & \Lambda^2(\mathbb{R}^3) \\ \wr & & \wr \\ & \xrightarrow{\text{curl}} & \end{matrix}$$

---

$$F'(x) : T_x M \to T_x N$$

$$\begin{matrix} M^k & \xrightarrow{F} & N^k \\ w & \alpha w = F^* \lambda & \lambda \\ & \alpha : M \to \mathbb{R} & \end{matrix}$$

$$(u_1 \ldots u_k) \qquad\qquad (w_1 \ldots w_k)$$

the change of basis $(F_* u_1 \ldots F_* u_k)$

from to $(w_1 \ldots w_k)$

has a pos. det.

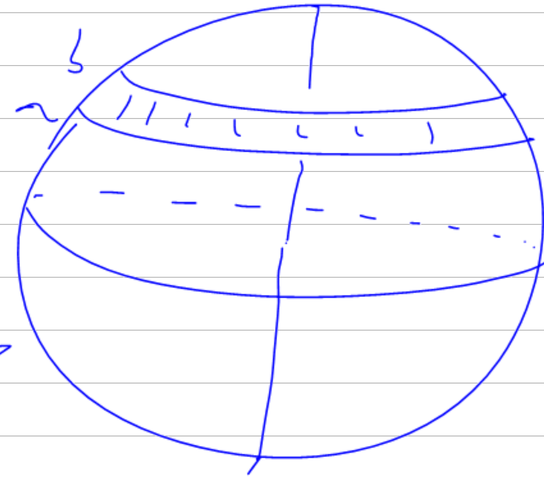
$$\begin{tikzcd} \mathbb{C} \arrow[r, "z = a+ib"] \arrow[d, equal] & \mathbb{C} \arrow[d, equal] \\ \mathbb{R}^2 \arrow[r, "\begin{pmatrix} a & -b \\ b & a \end{pmatrix}"] & \mathbb{R}^2 \end{tikzcd}$$

$$\det\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a^2 + b^2 \geq 0$$

---

$$w = \underbrace{\left( \frac{x\,dy - y\,dx}{x^2 + y^2} \right)} \wedge dz$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sqrt{1-z^2}\cos\theta \\ \sqrt{1-z^2}\sin\theta \\ z \end{pmatrix}$$

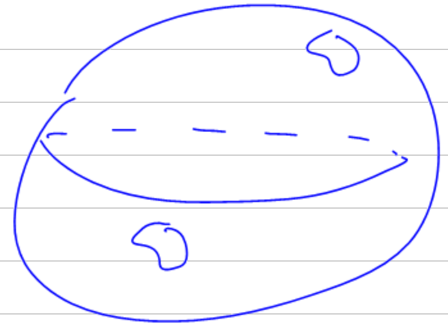



$$[0, 2\pi]_\theta \times [-1, 1]_z \xrightarrow{c}$$

$$c^* w = d\theta \wedge dz$$

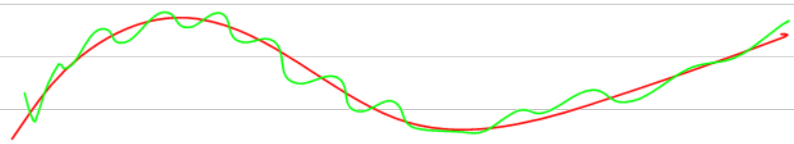
$$w = d\theta \wedge dz$$

$$\int_{[0,2\pi]_\theta \times [a,b]_z} w = 2\pi(b-a)$$

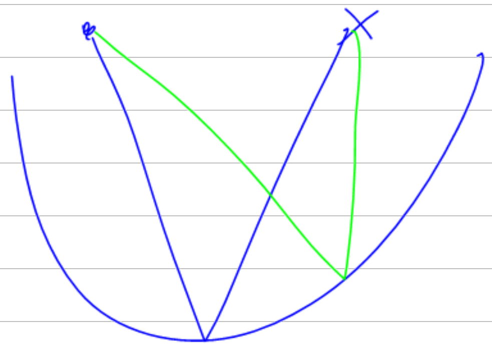


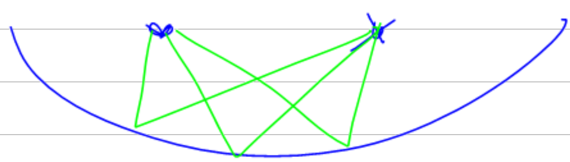
$F: \mathbb{R}^n \to \mathbb{R}^m$

$(\partial_i F_j) = F'$




Huygens principle





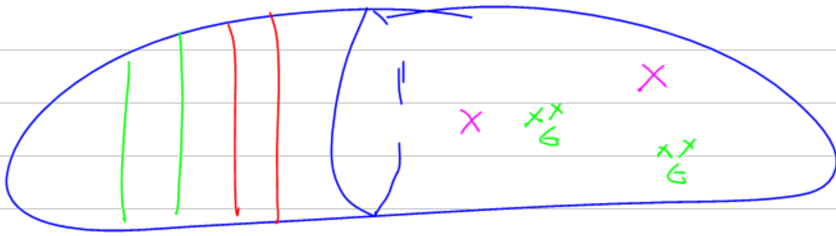
$$x\,dy\,dz + y\,dz\,dx + \underline{\underline{z\,dx\,dy}}$$

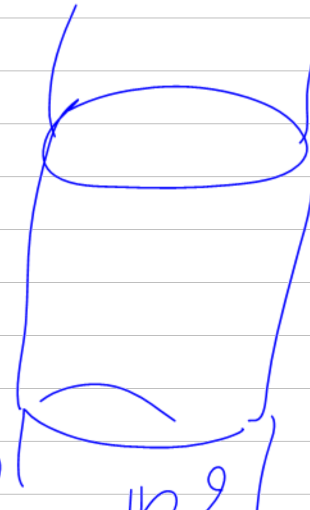
$$= \left(\frac{x^2}{z} + \frac{y^2}{z} + z\right) dx\,dy$$

$$= \left(\frac{1 - \cancel{z^2} + \cancel{z^2}}{z}\right) dx\,dy$$

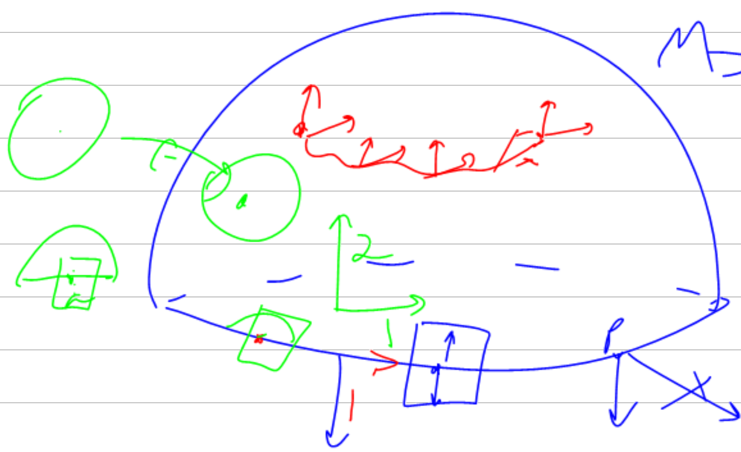
$$\frac{x\,dy - y\,dx}{x^2+y^2} \wedge dz = \frac{x^2/z + y^2/z}{x^2+y^2}\,dx\,dy$$

$$= \frac{1}{z}\,dx\,dy$$



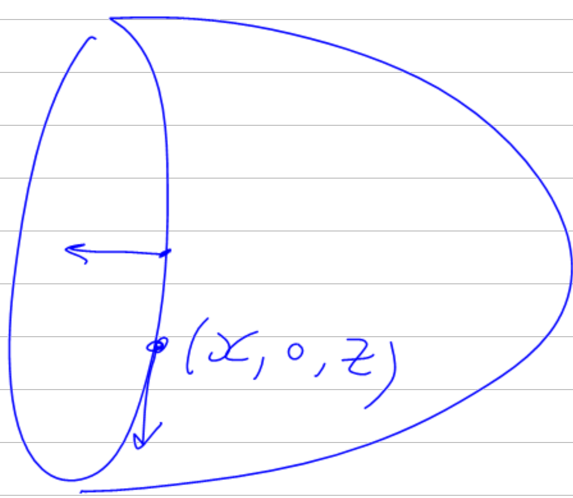


$$SO(3) = \left\{ A : \begin{array}{c} A^T A = I \\ \det A = 1 \end{array} \right\} \subset \mathbb{R}^9$$



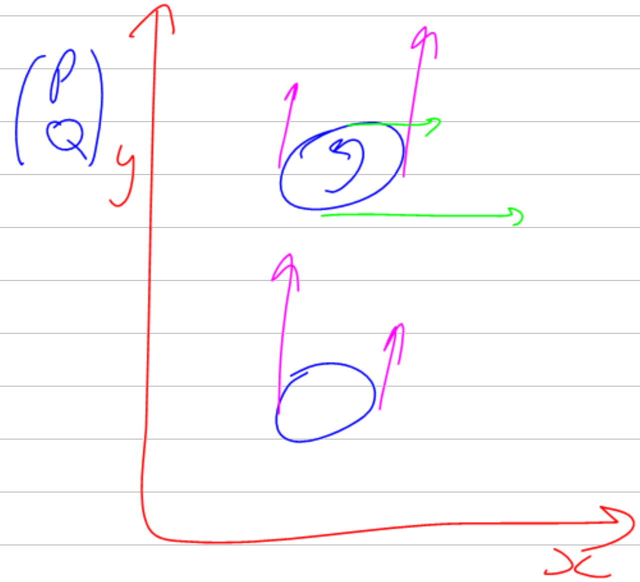
$$M = \left\{ (x,y,z) : \begin{array}{c} x^2+y^2+z=1 \\ z \geq 0 \end{array} \right\}$$

$$T_p M = \_ \quad \_ \; .$$

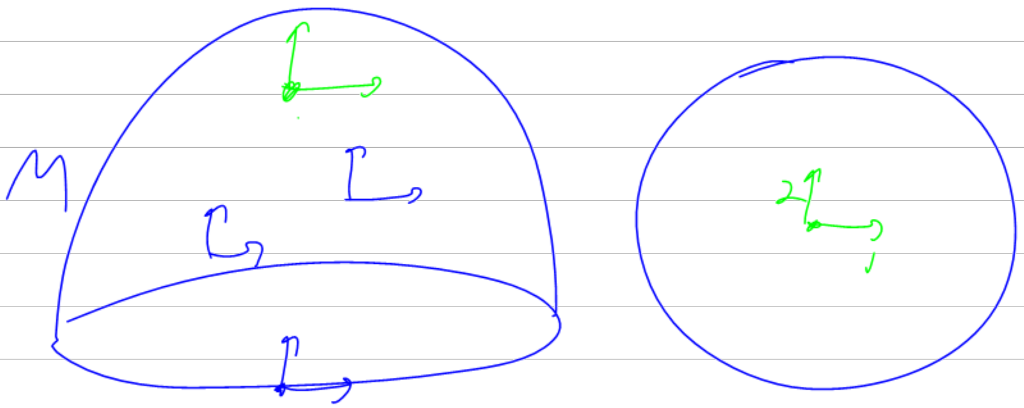



$$\xi = (p, v)$$

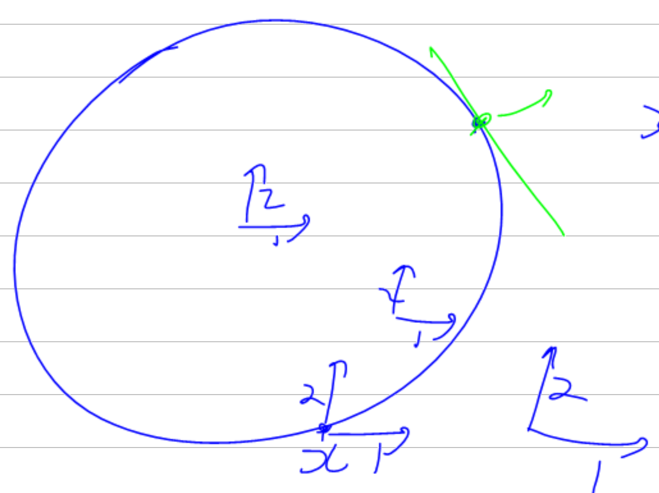
$$= ((x,0,z), (\underline{y}, \underline{z}, \underline{-x}))$$



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$



$$(e_1, e_2, e_3)$$



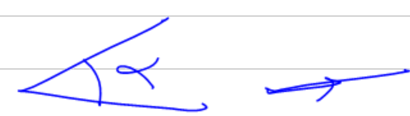
induce 1

$$x \in D^2 \subset \mathbb{R}^2$$

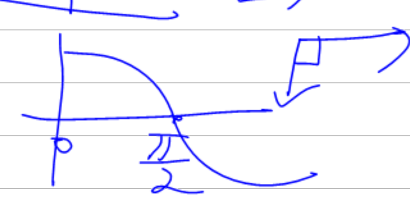
induce 2

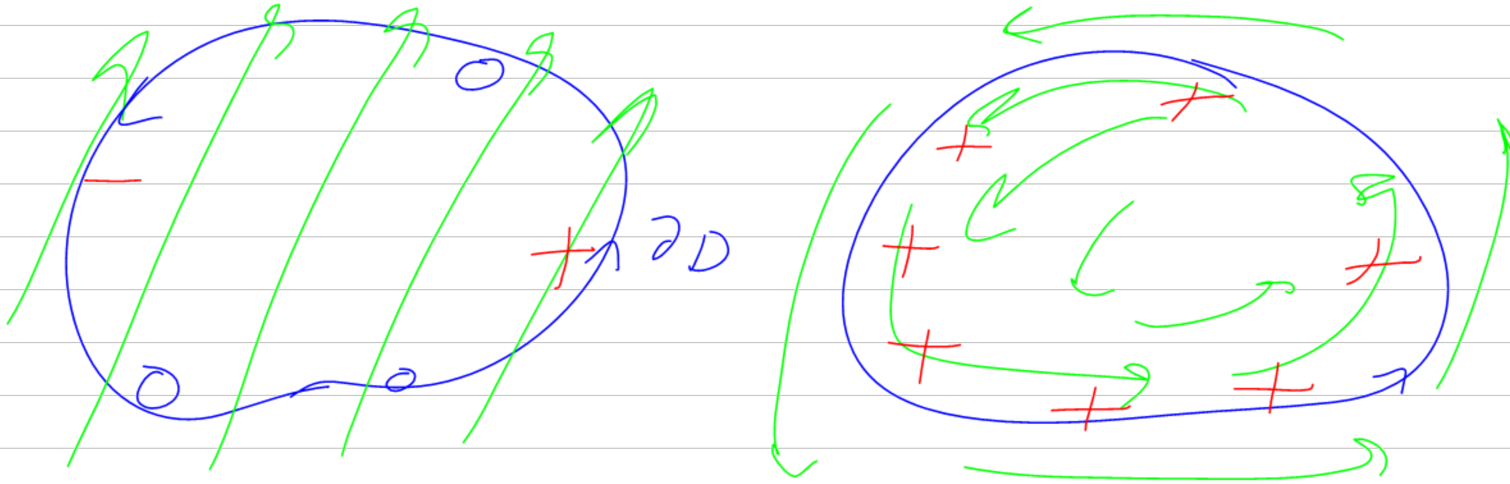
$$\in \partial D^2$$

$$V \cdot W = |V||W| \cos\alpha$$

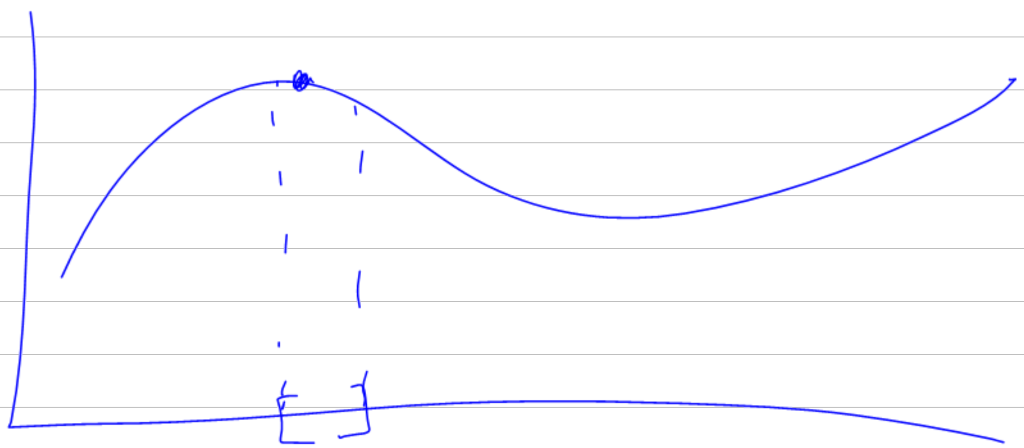


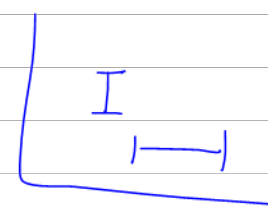
$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_{\partial D} \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \dot{\gamma}\, dt$$






$$F = \underline{\underline{B_x}}\, dy\, dz + \ldots$$
$$+ E_x\, dx\, dt$$






$$C_{I,n,\epsilon} : (t_1 \ldots t_k) \longmapsto a + \epsilon(0\ t_1\ 0\ t_2\ t_3\ \ldots\ )$$
$$\qquad\qquad\qquad\qquad 1\ 2\ 3\ 4\ 5\ \ldots\ n$$
$$\qquad\qquad\qquad\qquad\ \ i_1\quad i_2\ \ i_3$$

size

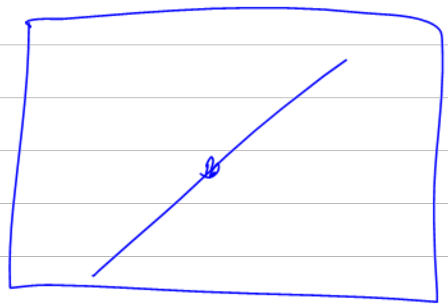
lowest corner

which coords are non-constant

$$= (i_1 \ldots i_k) \in \underline{n}_a^k \sim \binom{n}{k}$$

$\{1, \ldots, n\} = \underline{n}$ $\quad X^k = (x_1 \ldots x_k)$

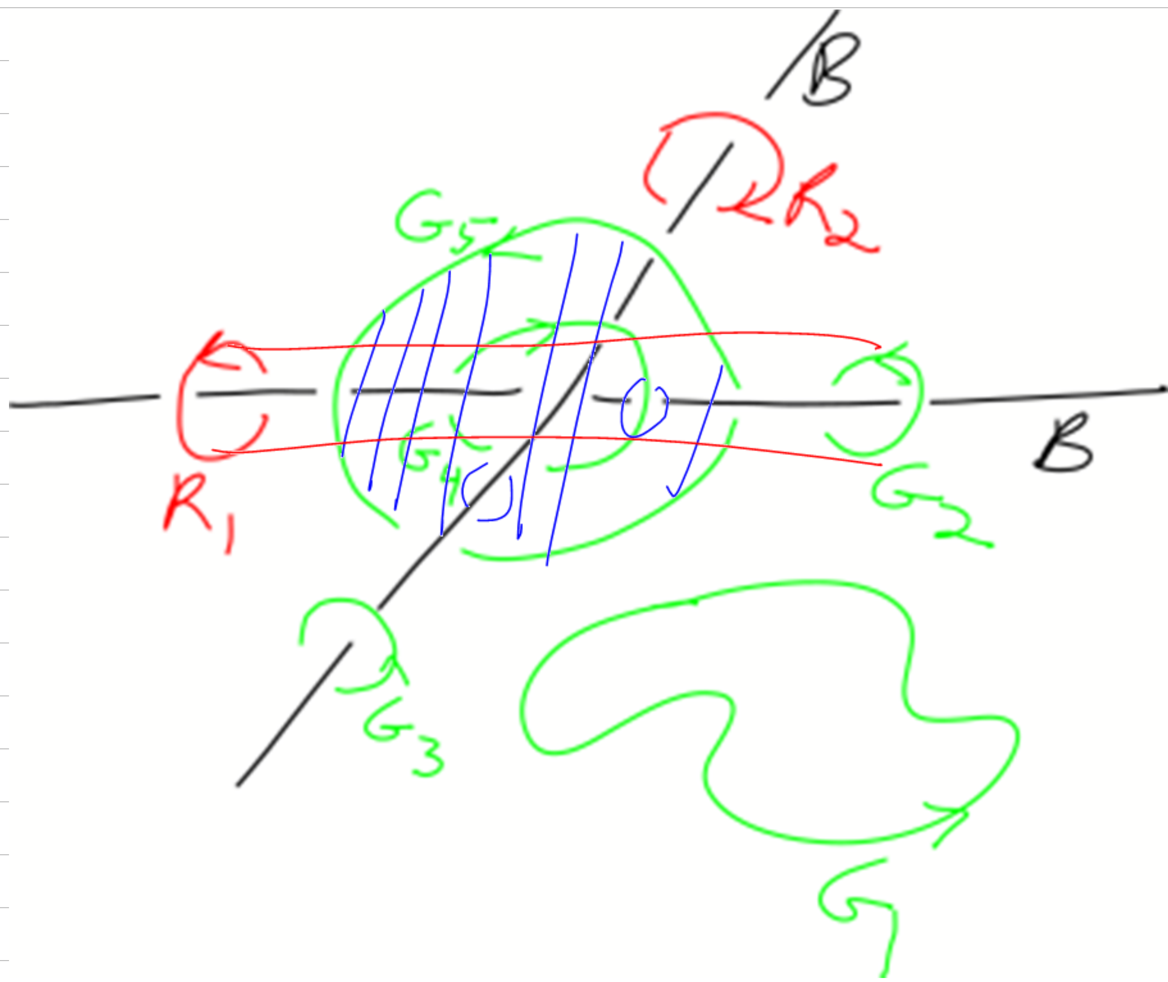
$\underline{n}^k = \{(i_1 \ldots i_k) : 1 \leq i_\alpha \leq n\}$

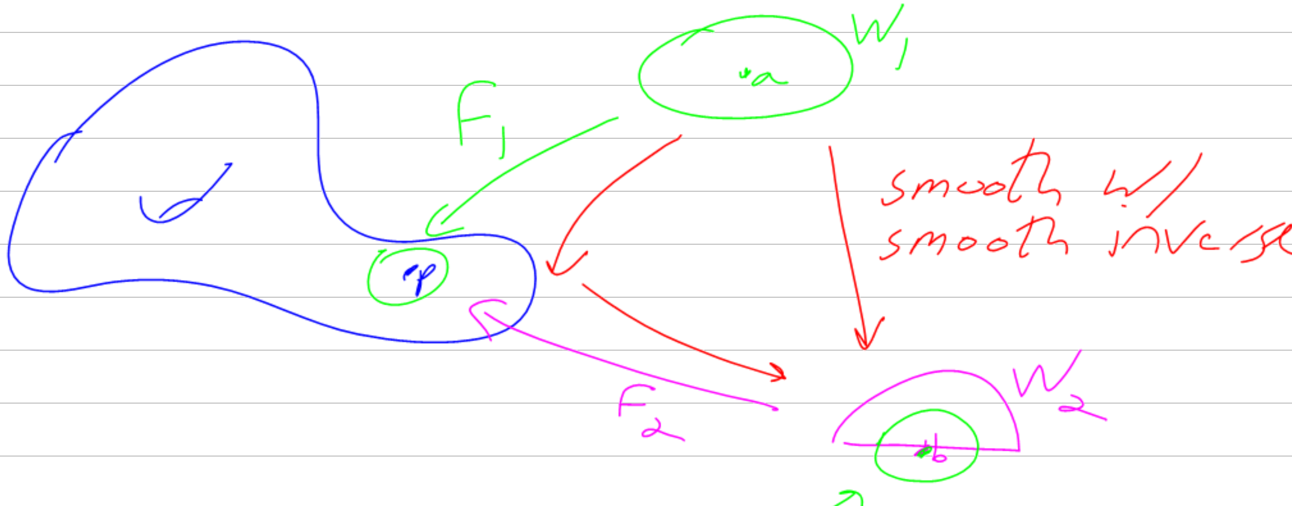


$|xy|^{1/2}$

$t \mapsto (t, t)$

$|tt|^{1/2} = |t|$






$$b \in \underbrace{F_2^{-1}(F_1(W_1))}_{\text{open}} \subset H_k$$

$$F(a+h) = F(a) + \underline{\underline{L}} \cdot h + \underline{\underline{o(h)}}$$

---

Prob 9 $\quad \mathbb{R}^n = \mathbb{R}^k_x \times \mathbb{R}^{n-k}_y \qquad F : \mathbb{R}^n \to \mathbb{R}^k$

wlog $\dfrac{\partial F}{\partial x}$ is invertible

$$\underbrace{\left( \begin{array}{c|c} k\times k & k\times(n-k) \end{array} \right)}_{n} \Big\} k$$

$$h(x,y) = \begin{pmatrix} F(x,y) \\ y \end{pmatrix}$$

$h : \mathbb{R}^n \to \mathbb{R}^n$

$$h' = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ 0 & I \end{pmatrix} \text{ is invertible near } 0$$

Set $g = h^{-1}$ (near 0)

$F(g(x,y))$
$\shortparallel$
$F(z,y)$
$\shortparallel$
$x$

$z$ s.t. $F(z,y) = x$
$g(x,y) = (z,y)$

$M^k \subset \mathbb{R}^n \quad \nu_1 \dots \nu_{n-k}$

$$dV(u_1 \dots u_k) = \begin{vmatrix} - & u_1 & - \\ & \vdots & \\ - & u_k & - \\ - & \nu_1 & - \\ & \vdots & \\ - & \nu_{n-k} & - \end{vmatrix}$$

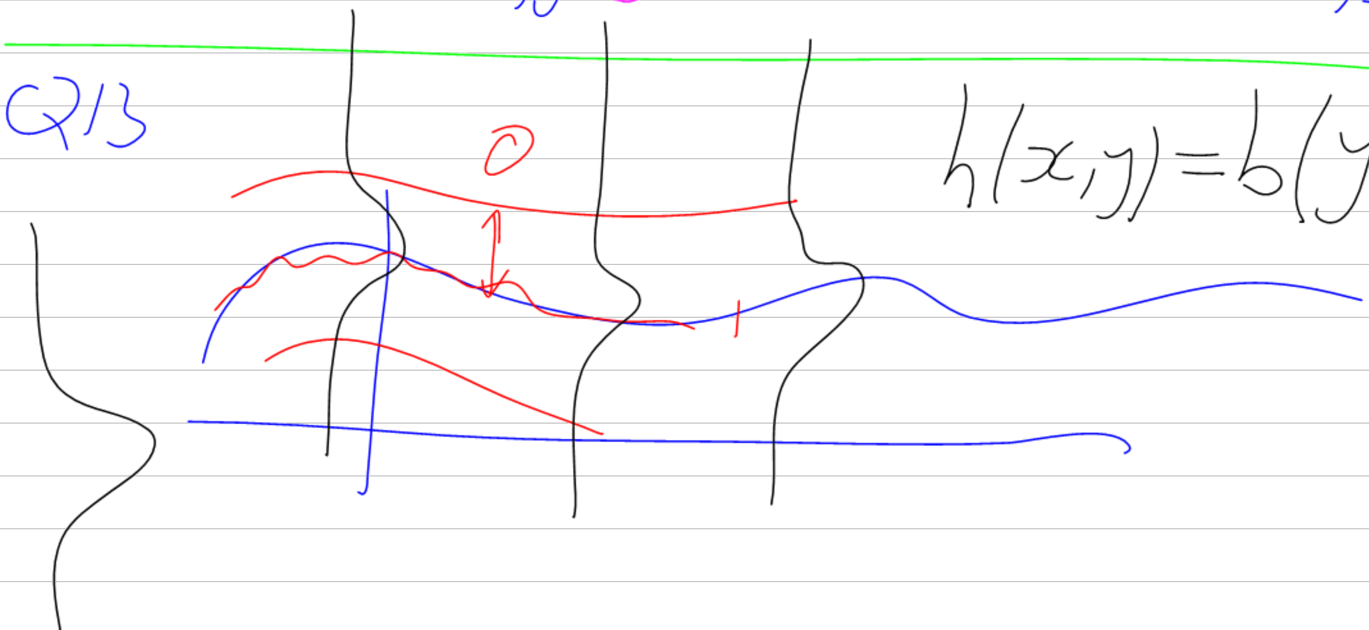
$\mathbb{R}^4_{t,xyz}$

| $\Omega^0$ | $\Omega^1$ | $\Omega^2$ | $\Omega^3$ | $\Omega^4$ |
|---|---|---|---|---|
| $f = W^0_f$ | $g\,dt + F_x\,dx + F_y\,dy + F_z\,dz = W^1_{g,F}$ | $G_x\,dt\,dx + G_y\,dt\,dy + G_z\,dt\,dz + H_x\,dy\,dz + H_y\,dz\,dx + H_z\,dx\,dy = W^2_{G,H}$ | $h\,dx\,dy\,dz + L_x\,dt\,dy\,dz + L_y\,dt\,dz\,dx + L_z\,dt\,dx\,dy = W^3_{h,L}$ | $l\,dt\,dx\,dy\,dz = W^4_l$ |

$$dW^0_f = W^1_{\partial_t f,\, \mathrm{grad}_3 f} \qquad dW^2_{G,H} = W^3_{\pm \mathrm{div} H,\, \pm \mathrm{curl}\, G \pm \partial_t H}$$

---

Q13




$$h(x,y) = b(y - f(x))$$

$b : \mathbb{R} \to \mathbb{R}$ ; $b(0) = 1$ $\quad b(x) = 0$ if $|x| \geq 1$

$b(x) \geq 0$

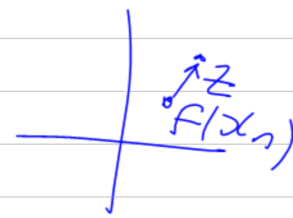
---

Q7 $\quad |(x-y) - (f(x) - f(y))| \leq \frac{1}{3}|x-y|$ $\quad f : \mathbb{R}^n \to \mathbb{R}^n \ni z$

$x_0 = 0$

$\vdots$

$x_n = x_{n-1} + (z - f(x_{n-1}))$


$x_{n+1} = x_n + (z - f(x_n))$

$$|x_{n+1} - x_n| = |(x_n - x_{n-1}) - (f(x_n) - f(x_{n-1}))|$$
$$\leq \frac{1}{3}|x_n - x_{n-1}| \leq \frac{1}{9}|x_{n-1} - x_{n-2}|$$
$$\leq \cdots \frac{1}{3^n}|x_1 - x_0|$$

$$|f(x_n) - z| = |x_{n+1} - x_n| \leq \frac{1}{3^n}|x_1 - x_0| \to 0$$

---

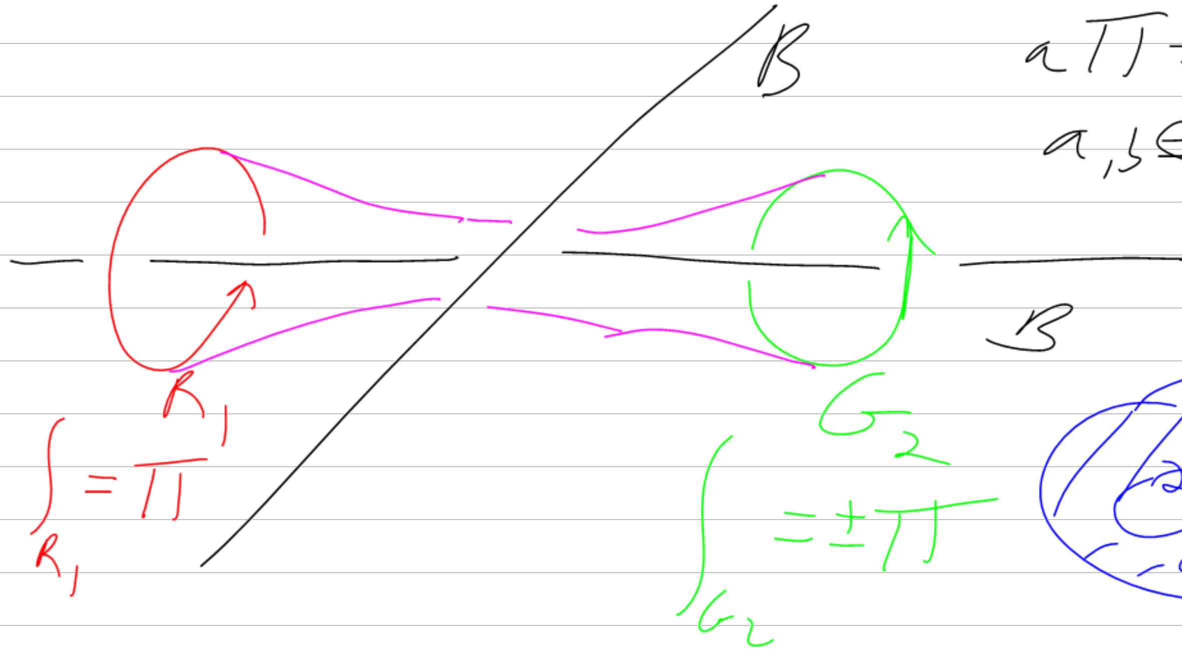
$\cancel{\langle x, x\rangle \geq 0 \quad = \iff x = 0}$

"$\langle\ ,\ \rangle$ is non-degenerate"
meaning $\quad \forall y\ \langle x, y\rangle = 0 \implies x = 0$

Greg Egan "orthogonal"

$a\pi + b\ell$

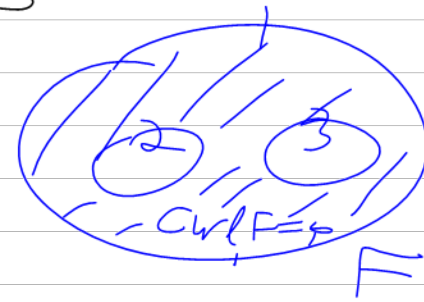
$a,b \in \{0, \pm 1\}$

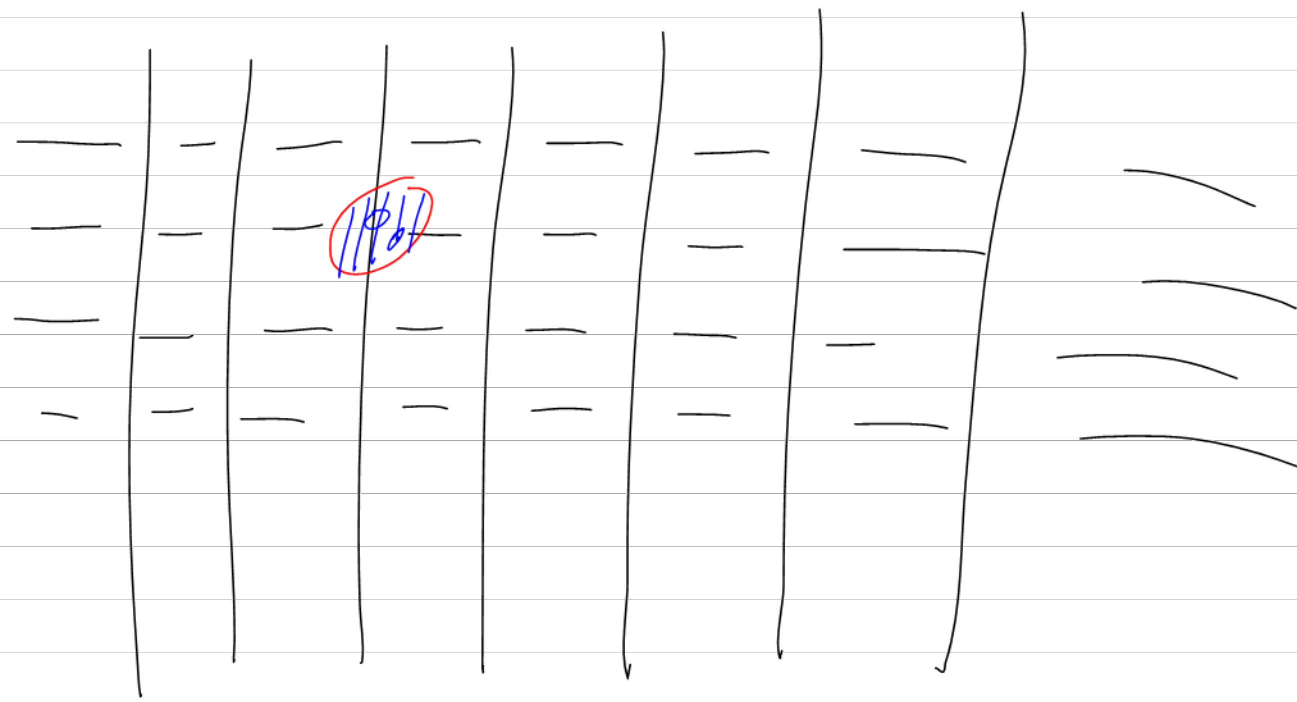



$\int_{R_1} = \pi$

$\int_{G_2} = \pm\pi$

$1 = 2 + 3$



Q19 Precise: $\int_C w$ depends only on $\partial C$

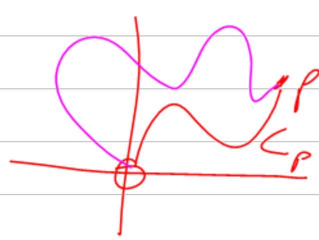
$w \in \Omega^1$ C: 1-cube.

Exact: $\exists \lambda$ s.t. $w = d\lambda$ $\lambda \in \Omega^0$

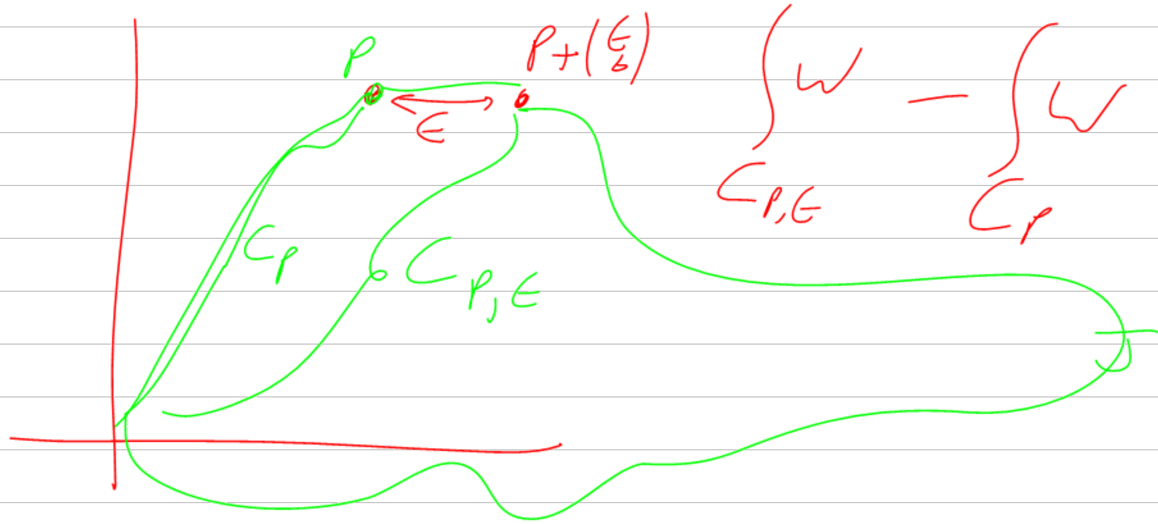
Precise $\Rightarrow$ Exact: wlog $\lambda(0) = 0$

$C_p(0) = 0$

$C_p(1) = p$




$\lambda(p) = \int_{C_p} w$




$$\int_{C_{p,\epsilon}} \omega - \int_{C_p} \omega = \int_p^{p+\binom{\epsilon}{0}} \omega$$


$$\sum_{i=1}^{\infty} \varphi_i F|_{W_i}$$