A Bit on Maxwell's Equations

There's also a handout at http://drorbn.net/2021-257/ap/Maxwell.pdf

Dror Bar-Natan: Classes: 2020-21: 2021-257 Analysis II:

A Bit on Maxwell's Equations

Prerequisites.

- Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega=0$, then there exists η with $d\eta=\omega$.
- Integration by parts: $\int_{\mathbb{R}^n}\omega\wedge d\eta=-(-1)^{\deg\omega}\int_{\mathbb{R}^n}(d\omega)\wedge\eta$ for compactly supported forms.
- The Hodge star operator $\star$ which satisfies $\omega\wedge\star\eta=\langle\omega,\eta\rangle dx_1\cdots dx_n$ whenever ω and η are of the same degree.
- The simplesest least action principle: the extremes of $q\mapsto S(q)=\int_a^b\left(\frac{1}{2}m\dot{q}^2(t)-V(q(t))\right)dt$ occur when $m\ddot{q}=-V'(q(t))$. That is, when $F=ma$.

Table 18–1 Classical Physics

Maxwell's equations

I. $\nabla\cdot E=\frac{\rho}{\epsilon_0}$ (Flux of E through a closed surface) = (Charge inside)/ϵ_0

II. $\nabla\times E=-\frac{\partial B}{\partial t}$ (Line integral of E around a loop) = $-\frac{d}{dt}$(Flux of B through the loop)

III. $\nabla\cdot B=0$ (Flux of B through a closed surface) = 0

IV. $c^2\nabla\times B=\frac{j}{\epsilon_0}+\frac{\partial E}{\partial t}$ c^2(Integral of B around a loop) = (Current through the loop)/ϵ_0 $+\frac{\partial}{\partial t}$(Flux of E through the loop)

[Conservation of charge
$\nabla\cdot j=-\frac{\partial\rho}{\partial t}$ (Flux of current through a closed surface) = $-\frac{\partial}{\partial t}$(Charge inside)]

Force law
$F=q(E+v\times B)$

Law of motion
$\frac{d}{dt}(p)=F$, where $p=\frac{mv}{\sqrt{1-v^2/c^2}}$ (Newton's law, with Einstein's modification)

Gravitation
$F=-G\frac{m_1m_2}{r^2}e_r$

The Feynman Lectures on Physics vol. II, page 18-2

The Action Principle. The *4-Vector Potential* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A):=\int_{\mathbb{R}^4}\frac{1}{2}\|dA\|^2\,dtdxdydz+J\wedge A}$$

where the 3-form J is the *charge-current*.

The Euler-Lagrange Equations in this case are $d\star dA=J$, meaning that there's no hope for a solution unless $dJ=0$, and that we might as well (think Poincaré's Lemma!) change variables to $F:=dA$. We thus get

$$\boxed{dJ=0\qquad dF=0\qquad d\star F=J}$$

These are the Maxwell equations! Indeed, writing $F=(E_xdxdt+E_ydydt+E_zdzdt)+(B_xdydz+B_ydzdx+B_zdxdy)$ and $J=\rho dxdydz-j_xdydzdt-j_ydzdxdt-j_zdxdydt$, we find:

$dJ=0\Longrightarrow$	$\operatorname{div}j=-\frac{\partial\rho}{\partial t}$	"conservation of charge"
$dF=0\Longrightarrow$	$\operatorname{div}B=0$	"no magnetic monopoles"
	$\operatorname{curl}E=-\frac{\partial B}{\partial t}$	that's how generators work!
$d*F=J\Longrightarrow$	$\operatorname{div}E=-\rho$	"electrostatics"
	$\operatorname{curl}B=j-\frac{\partial E}{\partial t}$	that's how electromagnets work!

Exercise. Use the Lorentz metric to fix the sign errors.

Exercise. Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers.

Exercise. With $ds^2=c^2dt^2-dx^2-dy^2-dz^2$ use $S=mc\int_{e_1}^{e_2}(ds+eA)$ to derive Feynman's "law of motion" and "force law".

Table 18–1 Classical Physics

Maxwell's equations

I.	$\nabla\cdot\boldsymbol{E} = \dfrac{\rho}{\epsilon_0}$	(Flux of $\boldsymbol{E}$ through a closed surface) $=$ (Charge inside)$/\epsilon_0$
II.	$\nabla\times\boldsymbol{E} = -\dfrac{\partial\boldsymbol{B}}{\partial t}$	(Line integral of $\boldsymbol{E}$ around a loop) $= -\dfrac{d}{dt}$ (Flux of $\boldsymbol{B}$ through the loop)
III.	$\nabla\cdot\boldsymbol{B} = 0$	(Flux of $\boldsymbol{B}$ through a closed surface) $= 0$
IV.	$c^2\nabla\times\boldsymbol{B} = \dfrac{\boldsymbol{j}}{\epsilon_0} + \dfrac{\partial\boldsymbol{E}}{\partial t}$	c^2(Integral of $\boldsymbol{B}$ around a loop) $=$ (Current through the loop)$/\epsilon_0$ $+\dfrac{\partial}{\partial t}$(Flux of $\boldsymbol{E}$ through the loop)

Conservation of charge

$$\nabla\cdot\boldsymbol{j} = -\frac{\partial\rho}{\partial t}$$

(Flux of current through a closed surface) $= -\dfrac{\partial}{\partial t}$ (Charge inside)

recover

Force law

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

Law of motion

$$\frac{d}{dt}(\boldsymbol{p}) = \boldsymbol{F}, \quad \text{where} \quad \boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}$$

(Newton's law, with Einstein's modification)

within easy reach

Gravitation

$$\boldsymbol{F} = -G\frac{m_1 m_2}{r^2}\boldsymbol{e}_r$$

The Feynman Lectures on Physics vol. II, page 18-2

Prerequisites.

- Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.
- Integration by parts: $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg \omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$ for compactly supported forms.
- The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.
- The simplesest least action principle: the extremes of $q \mapsto S(q) = \int_a^b \left(\frac{1}{2} m\dot{q}^2(t) - V(q(t)) \right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

Prerequisite 1.

Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.

Prerequisite 2.

$$\int f'g = -\int f g' \pm \text{bdry terms}$$

$$d(\omega\wedge\eta) = d\omega\wedge\eta \pm \omega\wedge d\eta$$

Integration by parts: $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg \omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$ for compactly supported forms.

$$0 = \int_{\partial B} \omega\wedge\eta = \int_{\mathbb{R}^n} d(\omega\wedge\eta) = \int_{\mathbb{R}^n} d\omega\wedge\eta \pm \int_{\mathbb{R}^n} \omega\wedge d\eta$$

Prerequisite 3.

$\binom{n}{k} = \binom{n}{n-k}$

$dx \longleftrightarrow dy \wedge dz$

$dy \longleftrightarrow dz \wedge dx$

~~$dx_I \longleftrightarrow dx_{I^c}$~~ naive.

The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.

$w \wedge (\ast w) = |w|^2 dx_1 \ldots dx_n$

Start from O.N. v_i consider the dual basis (φ_i)

$\langle \varphi_I, \varphi_J \rangle = \delta_{IJ}$

this is indep of the ON basis

Prerequisite 4.

The simplesest least action principle: the extremes of $q \mapsto S(q) = \int_a^b \left(\frac{1}{2}m\dot{q}^2(t) - V(q(t))\right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

The Action Principle.

The *4-Vector Potential* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} \left| dA \right|^2 dtdxdydz + J \wedge A}$$

where the 3-form J is the *charge-current*.

The Euler-Lagrange Equations

in this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

These are the Maxwell equations!

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d\star F = J}$$

Writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$dJ = 0 \Longrightarrow$	$\mathrm{div}\, j = -\frac{\partial \rho}{\partial t}$	"conservation of charge"
$dF = 0 \Longrightarrow$	$\mathrm{div}\, B = 0$	"no magnetic monopoles"
	$\mathrm{curl}\, E = -\frac{\partial B}{\partial t}$	that's how generators work!
$d * F = J \Longrightarrow$	$\mathrm{div}\, E = -\rho$	"electrostatics"
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$dJ = 0$

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

with $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$.

$$\boxed{dJ = 0 \Longrightarrow \quad \operatorname{div} j = -\frac{\partial \rho}{\partial t} \qquad \text{“conservation of charge”}}$$

$dF = 0$

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

with $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$.

$dF = 0 \Longrightarrow$	$\operatorname{div} B = 0$	"no magnetic monopoles"
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$d * F = J$

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

with $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and
$J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$.
With $\omega \wedge *\omega = |\omega|^2 dtdxdydz$ we have

$$*dxdt = -dydz, \qquad *dydt = -dzdx, \qquad *dzdt = -dxdy,$$

$$*dydz = -dxdt, \qquad *dzdx = -dydt, \qquad *dxdy = -dxdt,$$

so $*F = (-B_x dxdt - B_y dydt - B_z dzdt) + (-E_x dydz - E_y dzdx - E_z dxdy)$.

$d * F = J \Longrightarrow$	$\text{div}\, E = -\rho$	"electrostatics"
	$\text{curl}\, B = j - \dfrac{\partial E}{\partial t}$	that's how electromagnets work!

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III.	$\nabla \cdot \boldsymbol{B} = 0$	(Flux of $\boldsymbol{B}$ through a closed surface) = 0
IV.	$c^2 \nabla \times \boldsymbol{B} = \dfrac{\boldsymbol{j}}{\epsilon_0} + \dfrac{\partial \boldsymbol{E}}{\partial t}$	c^2(Integral of $\boldsymbol{B}$ around a loop) = (Current through the loop)/ϵ_0 $+ \dfrac{\partial}{\partial t}$ (Flux of $\boldsymbol{E}$ through the loop)
Conservation of charge		
	$\nabla \cdot \boldsymbol{j} = -\dfrac{\partial \rho}{\partial t}$	(Flux of current through a closed surface) = $-\dfrac{\partial}{\partial t}$ (Charge inside)
Force law		
	$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$	
Law of motion		
	$\dfrac{d}{dt}(\boldsymbol{p}) = \boldsymbol{F}$, where $\boldsymbol{p} = \dfrac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}$	(Newton's law, with Einstein's modification)
Gravitation		
	$\boldsymbol{F} = -G\dfrac{m_1 m_2}{r^2}\boldsymbol{e}_r$	

Feynman again. But wait, in our last two equations the sign of E is wrong!

Exercise 1.

Use the Lorentz metric to fix the sign errors.

Exercise 2.

Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers.

Exercise 3.

With $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ use $S = mc\int_{e_1}^{e_2}(ds + eA)$ to derive Feynman's "law of motion" and "force law".