Homework Assignment 20

Due: Monday April 12, 2021 11:59 PM (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Spivak's 5-25 (modified). We've shown that for a 2D manifold M in $\mathbb{R}^3$, we have that $dA = n_1 dx_2 \wedge dx_3 + n_2 dx_3 \wedge dx_1 + n_3 dx_1 \wedge dx_2$, where n is the positive unit normal to M. What replaces this formula if M is an $(n-1)$-dimensional manifold in $\mathbb{R}^n$?

Q2 (10 points)

Spivak's 5-26 (modified). (a) If $f\colon [a,b] \to \mathbb{R}$ is smooth and non-negative and the graph of $y = f(x)$ in the xy-plane is revolved around the x-axis in $\mathbb{R}^3$ to yield a surface M, show that the area of M is

$$\int_a^b 2\pi f \sqrt{1 + (f')^2}.$$

(b) Use this formula to compute the area of S^2.

(c) If a spherical loaf of bread is put into a bread cutting machine, which slice gets the most crust?

Q3 (10 points)

Spivak's 5-27 (modified). (a) Show that norm-preserving linear transformations of $\mathbb{R}^n$ into itself are also inner-product preserving.

(b) If T is such a transformation and M is a compact k-dimensional manifold in $\mathbb{R}^n$, show that the volume of M is the same as the volume of $T(M)$.

Q4 (10 points)

Spivak's 5-30 (modified). (a) Show that the length of the graph in $\mathbb{R}^2$ of a smooth function $f\colon [0,1] \to \mathbb{R}$ is given by $\int_0^1 \sqrt{1+(f')^2}$.

(b) Show that this length is the least upper bound of lengths of inscribed polygonal lines (finite polygonal lines connecting points on the graph of f heading left-to-right from $(0, f(0))$ to $(1, f(1))$. (Hint in text).

Q5 (10 points)

Spivak's 5-36 (modified). Let M be a compact 3D manifold sunk in the ocean, namely, contained in the lower half space $\mathbb{R}^3_{z\leq 0}$. The force the ocean exerts on M at a point $p \in \partial M$ is proportional to the water pressure, and hence to the depth $-z$ (ignoring atmospheric pressure), and it acts in the direction of $-n$, where n is the outward unit normal at p. Hence the force of buoyancy acting on M at the point p (the upward part of the force exerted by the fluid) is proportional to zn_3, and the overall force of buoyancy exerted on M is $\int_{\partial M} zn_3 dA$. Prove *Archimedes Law*, that this force is proportional to the amount of water displaced by M (namely, to the volume of M).

Riddle (closely related; do not submit, but you really want to know the answer, it's fun). Do you know a device so fine that it can tell that there is a difference in air pressure between the height of your chin and the height of your forehead?