Homework Assignment 15

Due: Monday March 1, 2021 11:59 PM (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Spivak's 4-19ab. If f is a function on $\mathbb{R}^3$ and F is a vector field on $\mathbb{R}^3$ with component functions F_1, F_2, F_3, define

$$\omega_F^1 := F_1 dx + F_2 dy + F_3 dz,$$

$$\omega_F^2 := F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy.$$

(a) Prove that

$$df = \omega^1_{\operatorname{grad} f},$$

$$d(\omega_F^1) = \omega^2_{\operatorname{curl} F},$$

$$d(\omega_F^2) = (\operatorname{div} F) dx \wedge dy \wedge dz.$$

(b) Use (a) to prove that

$$\operatorname{curl} \operatorname{grad} f = 0,$$

$$\operatorname{div} \operatorname{curl} F = 0.$$

(Note that the operators div, curl, and grad are defined in Spivak's text on pages 88 and 96).

Q2 (10 points)

Spivak's 4-20 (modified). Recall that a differential form ω is called "closed" if $d\omega = 0$, and "exact" if there exist another differential form λ such that $\omega = d\lambda$. Suppose the open sets $U, V \subset \mathbb{R}^n$ are diffeomorphic (meaning that there is a differentiable $g: U \to V$ that has a differentiable inverse $g^{-1}: V \to U$). Suppose it is known that every closed form on U is exact. Show that the same is true on V.

Q3 (10 points)

Spivak's 4-21 (modified). Let $\theta: \mathbb{R}^2_{x,y} \to \mathbb{R}$ be the function that assigns to every point (x, y) its "angle" θ when it is written using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Note that in order to make θ well-defined we need to exclude some ray from the x, y plane; for example, we can exclude the ray $\{(x, 0) : x \leq 0\}$. Prove that where θ is defined, we have

$$d\theta = \frac{-y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy.$$

Q4 (10 points)

Munkres' 30-2 Consider the forms

$$\omega = xydx + 3dy - yzdz,$$

$$\eta = xdx - yz^2dy + 2xdz,$$

in $\mathbb{R}^3$. Verify by direct computations that $d(d\omega) = 0$ and that $d(\omega \wedge \eta) = (d\omega) \wedge \eta - \omega \wedge (d\eta)$.

Q5 (10 points)

Munkres' 30-6 (modified). Often in mathematical notation, we put a hat on top of elements in a sequence that we wish to omit. For example, $(1 \ldots \hat{4} \ldots 7)$ means (123567). With this in mind, let ω be the form

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{|x|^p} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

in $\Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$, where p is some positive real number.

(a) Compute $d\omega$.

(b) For what value of p is $d\omega = 0$?