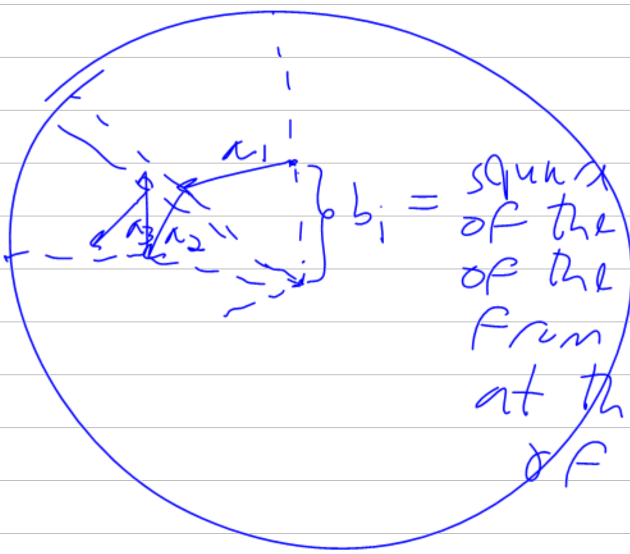


b_i = square of the distance of the student from the center at the beginning of step #i

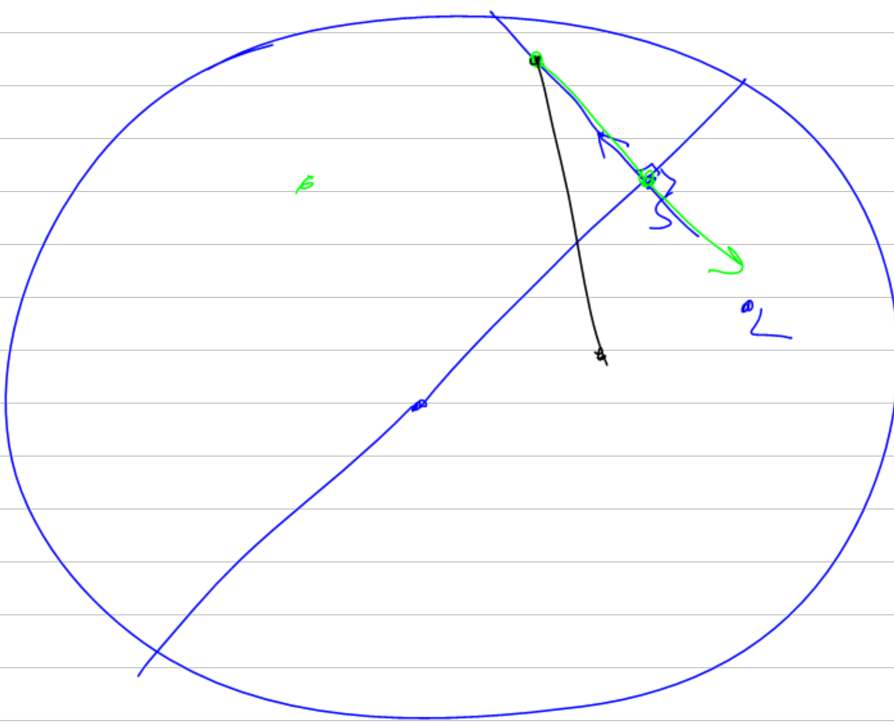
$$b_{i+1} = b_i + a_i^2$$

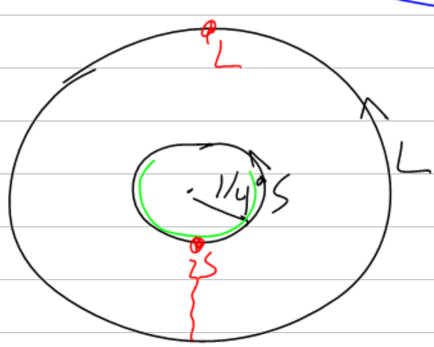
$$b_{n+1} = b_0 + \sum_{i=1}^{n} a_i^2 < 1$$

$$b_0 + \sum a_i^2 < 1$$

$$\sum a_i = \infty$$

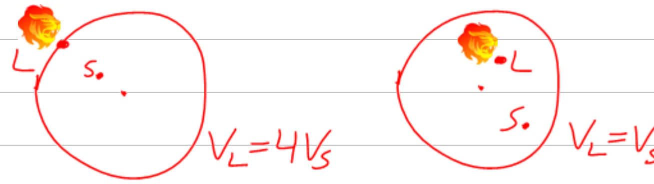
$$a_k = \frac{1}{k+c}$$

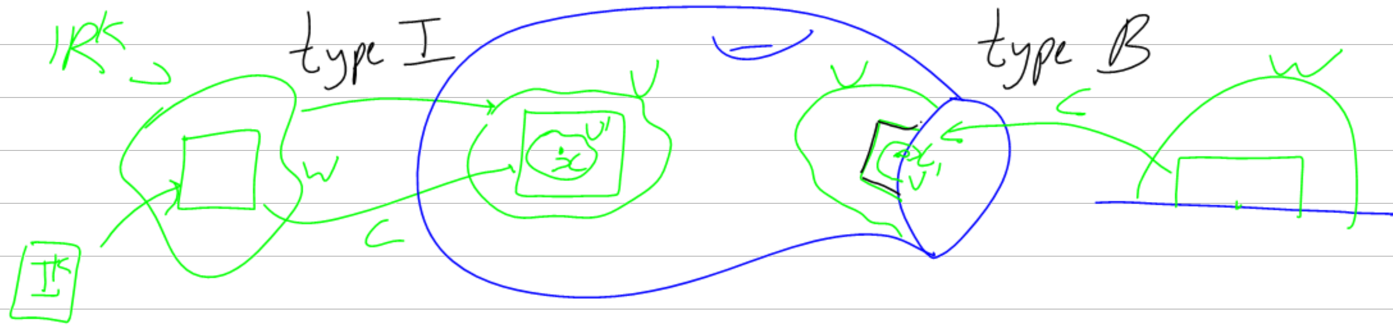




$$0.75 < \frac{\pi}{4}$$

Hour 67 MAT257 Analysis II on March 29, 2021: Stokes' Theorem.
Read Along: Spivak 122-126.
HW18 due and HW19 assigned by 11:59PM.
Old riddles (sol'n at end). Survive!





$\int \omega := \sum \int \varphi_i \omega$

Stokes' Thm If M is compact and oriented & $\omega \in \Omega^{k-1}(M)$, then $\int_M d\omega = \int_{\partial M} \omega$

Pf for type B $\omega \in \Omega^{k-1}(M)$, supp $\omega \subset U'$ (type B)

$$\int_M d\omega = \int_C d\omega = \int_{I^k} c^* d\omega = \int_{I^k} d(c^*\omega) = \int_{\partial I^k} c^*\omega$$

$$= -\int_{\substack{I^k_{(1,0)} \\ (y_1 \cdots y_{k-1})}} c^*\omega = + \int_{\partial M} \omega$$

Pf of Thm (assembly)

$$\int_{\partial M} \omega = \sum_i \int_{\partial M} \varphi_i \omega = \sum_i \int_M d(\varphi_i \omega)$$

$d\varphi_i \wedge \omega + \varphi d\omega$

$$= \sum_i \int_M d\varphi_i \wedge \omega + \sum_i \int_M \varphi_i \, d\omega$$

$$\int_M d\Big(\sum_i \varphi_i\Big) \wedge \omega = \int_M \big(d\,1\big) \wedge \omega = 0$$

$$\int_M d\omega$$

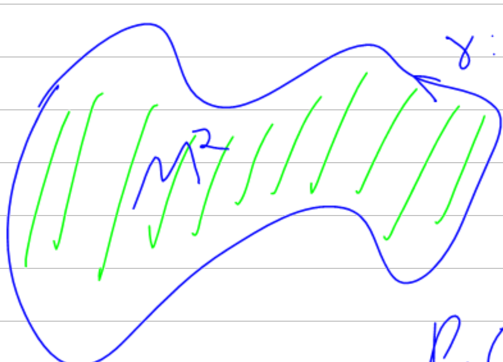
$\square$

<u>Example</u> $M = [a,b]$ $\quad \omega = f \in \Omega^0([a,b])$

$$\int_a^b f'\,dx = \int_{[a,b]} df = \int_{\partial[a,b]} f = f(b) - f(a)$$

Fund thm of calculus.

$\gamma : [0,1] \to \mathbb{R}^2$



$\subset \mathbb{R}^2$

M : connected simply connected domain in $\mathbb{R}^2$ w/ bdry a smooth curve γ.

$P, Q : M \to \mathbb{R}$

$$\omega = P\,dx + Q\,dy \in \Omega^1(M)$$

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

$$\int_M \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_\gamma P\,dx + Q\,dy$$

$t \in [0,1]$ $\gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix}$

$$= \int_{[0,1]} \gamma^*\left(P\,dx + Q\,dy\right)$$

$dx = d\gamma_1(t)$
$= \gamma_1'(t)\,dt$

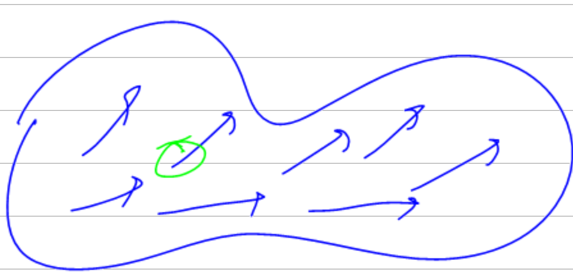
$$= \int_{[0,1]} P(\gamma(t)) \cdot \gamma_1'(t)\,dt + Q(\gamma(t)) \cdot \gamma_2'(t)\,dt$$

$$= \int_{[0,1]} \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \underbrace{\begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix}}_{\dot\gamma} dt = \int_{[0,1]} \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \dot\gamma\, dt$$

$$\int \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \dot\gamma\, dt = \int_M \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

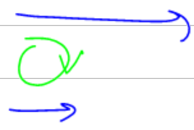
Two interpretations:

"curl": $\begin{pmatrix} P \\ Q \end{pmatrix}$: a vector field on M



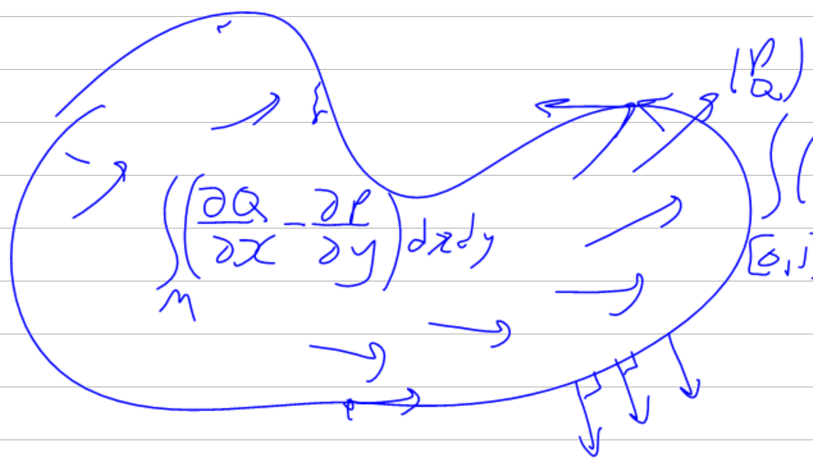
$$\underbrace{\boxed{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}}_{\text{Rotational force}} = ?$$

$$= \frac{\partial Q}{\partial x}$$



$$= \frac{\partial P}{\partial y}$$

experienced by a small cork disk floating on fluid that flows like $\begin{pmatrix} P \\ Q \end{pmatrix}$.

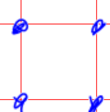


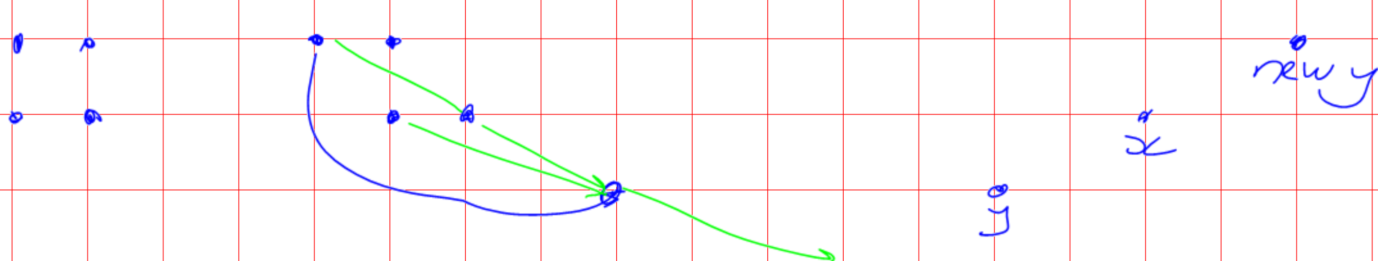
$$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$\int_{[0,1]} \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \dot\gamma \, dt$$

$\parallel$

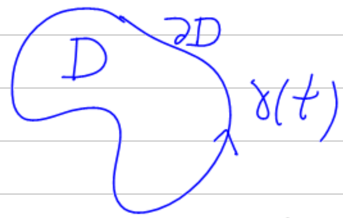
Force exerted on a rope γ by the flow of the fluid.




new y
x
y

Hour 68 MAT257 Analysis II on March 31, 2021: Green's Theorem, the planimeter.
Read Along: Spivak 122-126.
No class Friday!
I'm giving a talk in Moscow (ha) tomorrow. Wish me luck!
Old riddles (sol'n at end). Four points form a perfect square in the plane. Can you turn that square into a larger one by a sequence of moves of the form (x,y)->(x,2x-y) performed on pairs from within our four points? (Namely, by reflections of y about x?)

Green's theorem.

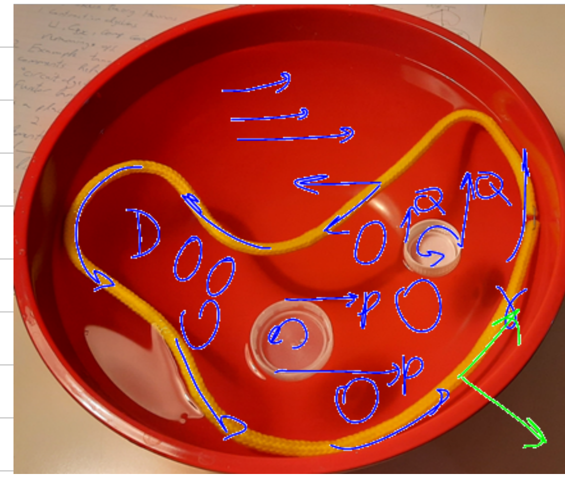


$$\omega = P\,dx + Q\,dy$$

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_0^1 \dot\gamma \cdot \begin{pmatrix} P \\ Q \end{pmatrix} dt$$

If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, measuring area.



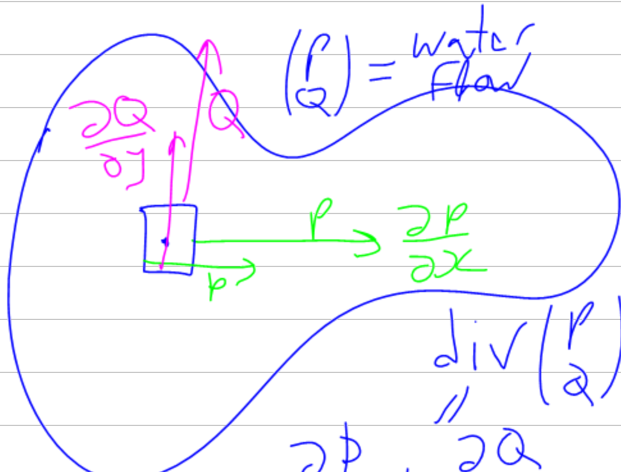
$$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{90°} \begin{pmatrix} +b \\ -a \end{pmatrix}$$ clockwise rotation

2nd interpretation $Q \longrightarrow P \qquad P \longrightarrow -Q$

$$\int_D \operatorname{div}\begin{pmatrix} P \\ Q \end{pmatrix} = \int_D \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)}_{\operatorname{div}\begin{pmatrix} P \\ Q \end{pmatrix}} = \int_0^1 \begin{pmatrix} \dot\gamma_1 \\ \dot\gamma_2 \end{pmatrix} \cdot \begin{pmatrix} -Q \\ P \end{pmatrix} dt = \int_0^1 \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \begin{pmatrix} +\dot\gamma_2 \\ -\dot\gamma_1 \end{pmatrix} dt$$

$$= \int_0^1 \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \vec{n}\, dt$$

$\vec{n}$: outgoing normal



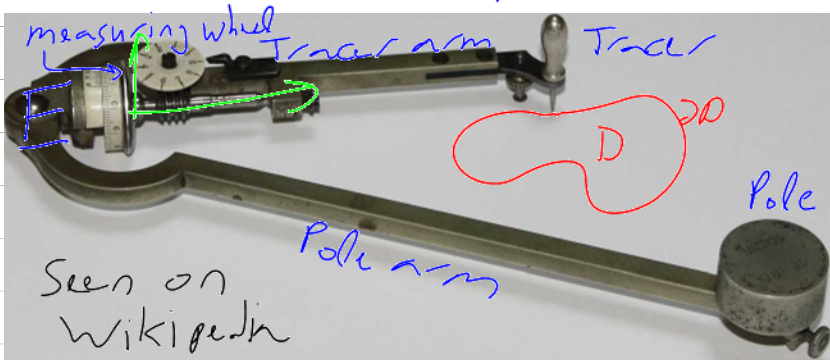
$\begin{pmatrix} P \\ Q \end{pmatrix}$ = water flow

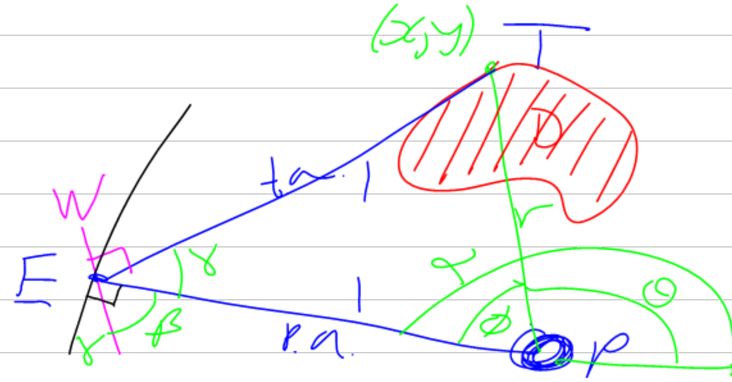
$\operatorname{div}\begin{pmatrix} P \\ Q \end{pmatrix}$ = how much water is inserted at every point.

$$\operatorname{div}\begin{pmatrix} P \\ Q \end{pmatrix} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

$$\overbrace{\text{Amount of water inserted over all of } D = \text{amount of water exiting over } \partial D}^{\text{conservation of water}}$$

Example. The planimeter





M = configuration space of the planimeter $\subset \mathbb{R}^2_{\alpha,\gamma}$

$\subset \mathbb{R}^2_{x,y}$

$\subset \mathbb{R}^2_{r,\theta}$

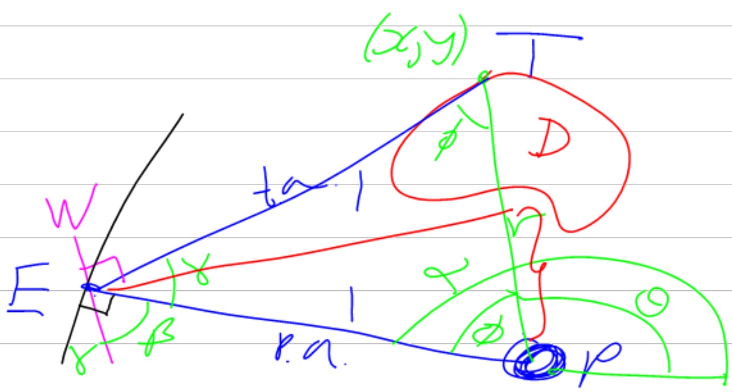
$\cap$

$\mathbb{R}^4_{\alpha,\gamma,x,y}$

$M \to \gamma, \beta, r, x$

$W : \begin{Bmatrix}\text{infinitesimal} \\ \text{motions}\end{Bmatrix} \to$ speed of turning of measuring wheel.

measurement process $= \int_{\text{path of motion}} W$



$x = r\cos\theta$

$y = r\sin\theta$

$dx \wedge dy = r\, dr \wedge d\theta$

$\alpha = \theta + \phi$

$\gamma = \pi - 2\phi$

$r = 2\cos\phi$

$$W = d\alpha \cdot \cos\gamma = d(\theta+\phi)\cdot\cos(\pi - 2\phi)$$

$$= -\cos(2\phi)\cdot(d\theta + d\phi)$$

$$dW = 2\sin 2\phi \, d\phi \wedge d\theta$$

$\sin 2\phi = 2\sin\phi\cos\phi$

$$= \underbrace{2\cos\phi}_{r} \cdot \underbrace{2\sin\phi\, d\phi}_{dr} \wedge d\theta = r\, dr \wedge d\theta$$

$$= dx \wedge dy$$

$$\oint_{\partial D} \omega = \int_D dx \wedge dy = \text{Area}(D).$$