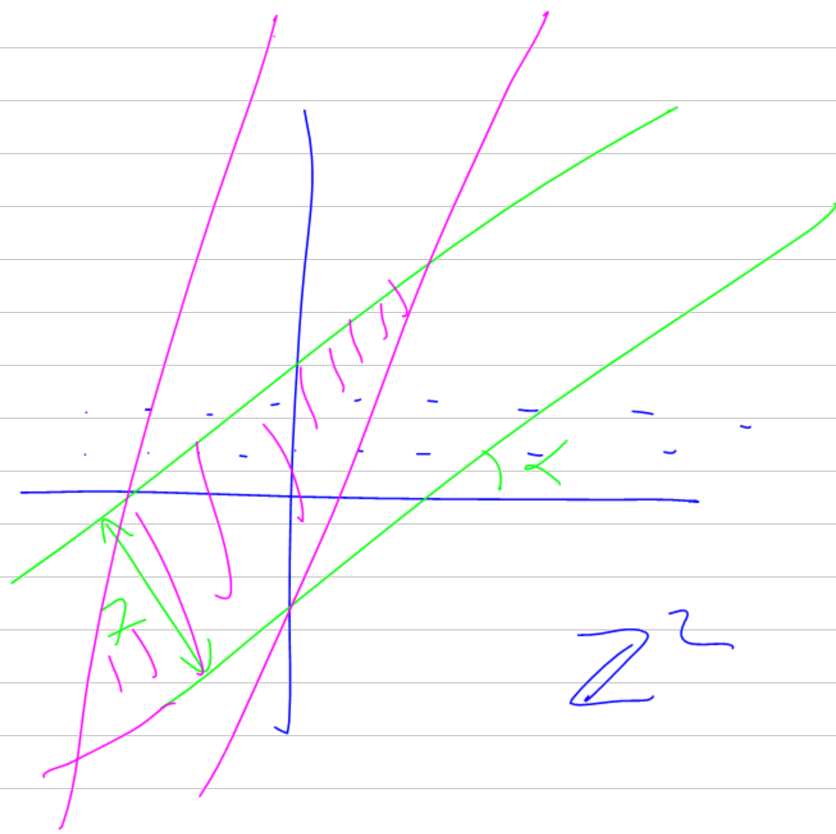


$\mathbb{Q} \hookrightarrow \mathbb{R}$

$\{r_n\}$ $\quad r_n \xrightarrow[n\to\infty]{} x$

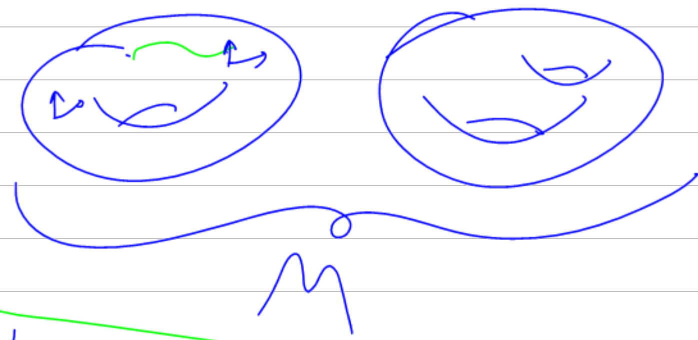
$\mathbb{Z}^2$

Hour 64 MAT257 Analysis II on March 22, 2021: Orientation, integration. → signtology.
Read Along: Spivak 117-125.
HW17 due on Wednesday, HW18 on web by midnight.
Old riddle (sol'n at end). Can you find uncountably many sets of natural numbers, such that the intersection of any two of them is finite?

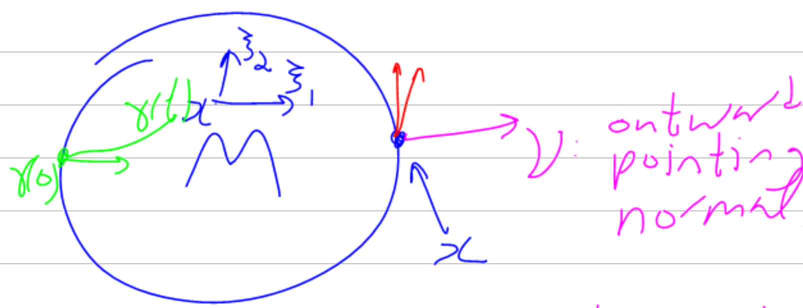
Orientation of $M^k$: A choice of an orientation for each $T_xM$, which can be presented in a locally ~~continuous~~ constant manner.
"orientable": has an orientation. "oriented": comes with one.

Comment A connected mfd has either 0 or 2 orientations.

Suppose $M^k$ is an oriented mfd w/ bdry, orient the bdry of $M$, $\partial M$, as follows




Suppose $x \in \partial M$ orient $T_x(\partial M)$ by choosing t.v. $\eta_1 \dots \eta_{k-1}$ in $T_x(\partial M)$



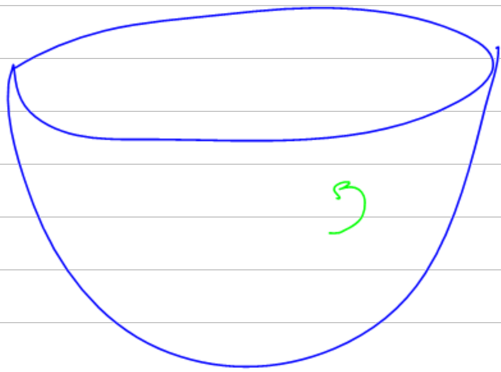

$T_x(M) \ni \nu \perp T_x(\partial M)$ in $\mathbb{R}^n$

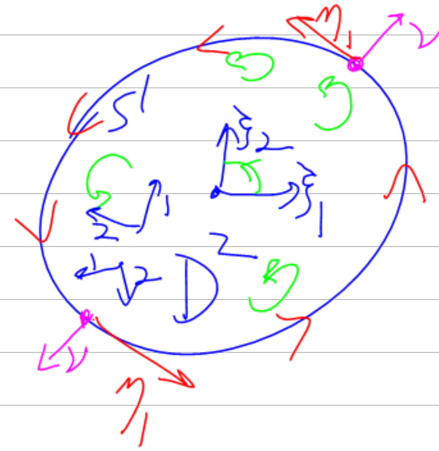
s.t. $\nu, \eta_1 \dots \eta_{k-1}$ is the given orientation of $T_x(M)$

Claim (LinAlg) This is well-defined!

Examples ② $M = D^2 = \{x \in \mathbb{R}^2 : |x| \leq 1\}$

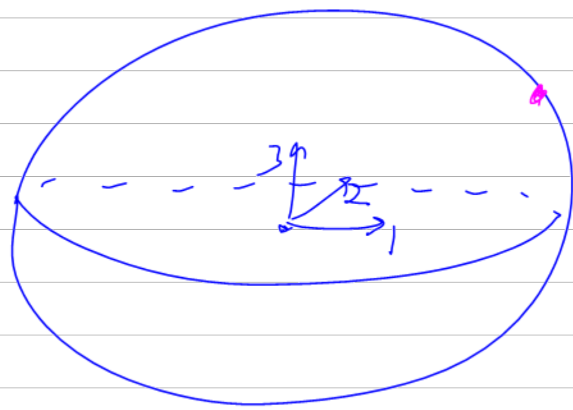
$\partial D^2 = S^1$:

 $\subset \mathbb{R}^3$



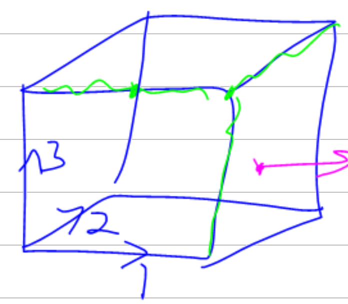

(3) $D^3 = \{x \in \mathbb{R}^3 : |x| \le 1\}$ $\quad \partial D^3 = S^2$

oriented using the "std orientation of $\mathbb{R}^3$" $(e_1, e_2, e_3)$



$\Longrightarrow$ $\partial D^3 = S^2$ is oriented as last time.

(K) $I^k$ oriented in the std manner $(e_1 \dots e_k)$



$I_{(j,\alpha)}$ has two orientation that come to mind:

(a) As a part of $\partial I^k$

(b) By identifying $I_{(j,\alpha)}$ w/ $I^{k-1}$ via

$I^{k-1} \ni y \mapsto (y_1, \dots \overset{j}{\alpha}, \dots y_{k-1}) \in I_{(j,\alpha)}$

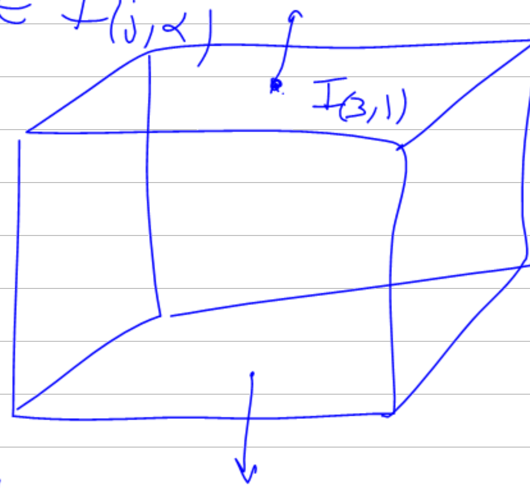



Figure out the two orientations on $I_{(j,\alpha)}$:

$O_b : (e_1, \dots, e_{j-1}, e_{j+1}, \dots e_k)$

this is consistent w/ $O_a$ if when we prepend $\nu$ to it, we get $(e_1, \dots e_k)$ of $I^k$

$$\nu = -(-1)^\alpha e_j = \begin{cases} +e_j & \alpha = 1 \\ -e_j & \alpha = 0 \end{cases}$$

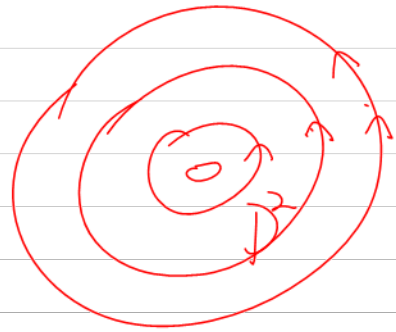
$$(-(-1)^\alpha e_j, e_1, \dots e_{j-1}, e_{j+1} \dots e_k) \overset{?}{\sim} (e_1 \dots e_k)$$

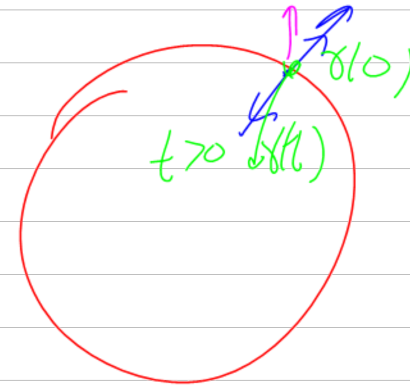
differ by a sign $s$,

$$s = -(-1)^\alpha \cdot (-1)^{j-1} = (-1)^{j+\alpha}$$

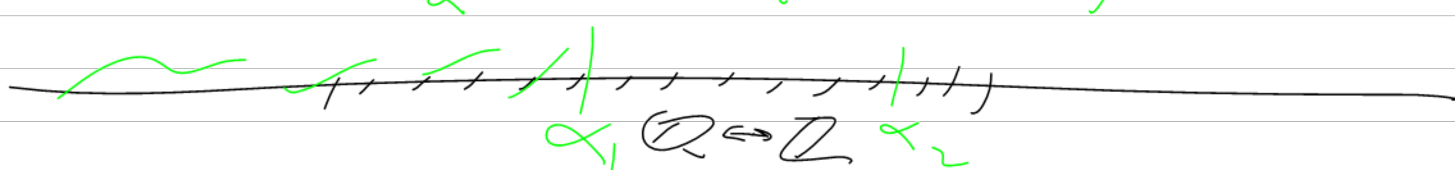
That's why we've defined

$$\partial c = \sum_{j,\alpha} (-1)^{j+\alpha} c_{(j,\alpha)}$$



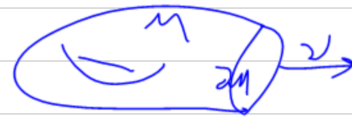



$$A_\alpha := \mathbb{Q} \cap (-\infty, \alpha)$$



Hour 65 MAT257 Analysis II on March 24, 2021: Orientation, integration.
Read Along: Spivak 117-125.
HW17 is due tonight, HW18 is on web.
Old riddle (sol'n at end). Can you find uncountably many sets of natural numbers, such that for any two of them, one contains the other?

Suppose $x \in \partial M \subset \underset{\text{oriented}}{\overset{\text{given}}{M}}$ and $\nu \in T_xM$ is the outward-pointing normal to $T_x\partial M$

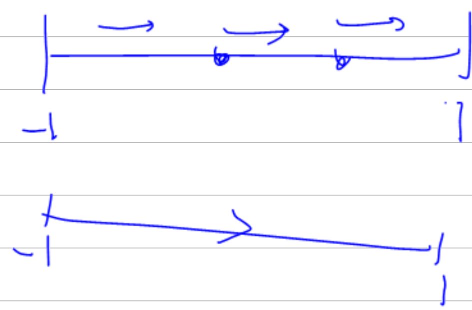



then $\mathcal{O}_{T_xM} = [\nu, u_1, \dots, u_{k-1}]$ $\quad \mathcal{O}_{T_x\partial M} = [u_1, \dots, u_{k-1}]$

(2) $D^2$ (3) $D^3$ (k) $I^k$ (1) $D^1 \sim I^1$

$D^1 = [-1,1] \subset \mathbb{R}$, $\quad I^1 = [0,1]$

$\partial D^1 = \{-1, +1\}$ 0-dim mfld.




An orientation at $x \in M^k$ is also $\eta_x \in \Lambda^k(T_xM) \big/ \text{mult by pos scalar}$.


Tools: Can use $\nu$

2. Can use $i^*$: Forms on $M$ $\to$ Forms on $\partial M$ $\quad i: \partial M \hookrightarrow M$

Aside Given a vector field $X$, define an op:

$$\iota_X : \Omega^*(M) \longrightarrow \Omega^{*-1}(M)$$

"interior multiplication by $X$"

by $(\iota_X \underset{p\text{-form}}{w})(u_1, \dots, u_{p-1}) = w(X, u_1, \dots, u_{p-1})$

* well-defined
* linear in $w$
* linear in $X$: $\iota_X + \iota_Y = \iota_{X+Y}$

<u>Exercise</u> Suppose $M^k$ is oriented, $\partial M$ has the induced orientation, $\nu$ is the outward pointing normal, $\eta_x \in \Lambda^k(T_x M)$ represents the orientation of $M$ & $\lambda_x \in \Lambda^{k-1}(T_x \partial M)$ represents the orientation of $\partial M$, then for $x \in \partial M$,

$$\lambda_x \text{ " = " } i^*(\iota_\nu \eta_x)$$ up to multiplication by a pos scalar.

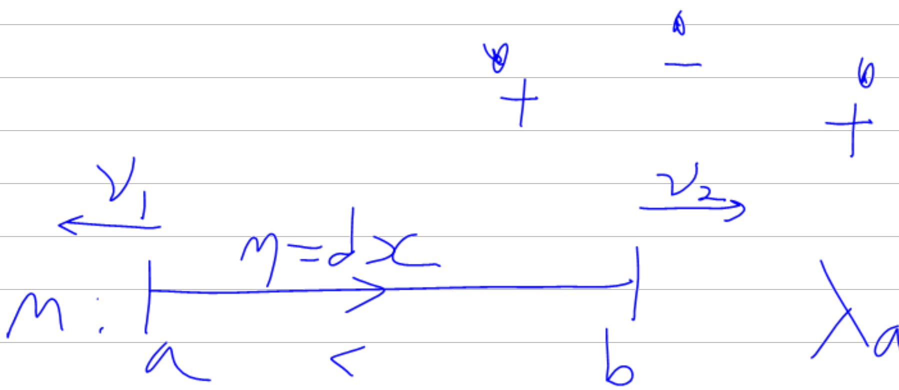
$$\eta \sim [v_1 \ldots v_k] \iff \eta(v_1 \ldots v_k) > 0$$

---

What is an orientation of a single pt?

<u>Ans</u> $\eta \in \Lambda^0(T_{pt}\, pt)$ $= $ a sign $+$ or $-$

pos scalars




$$\lambda_a = i^* \iota_{\nu_1}(dx)$$
$$= i^* dx(\nu_1) = -1$$

$$\partial M = \{a, b\}$$

$$\lambda_b = \ldots\ dx(\nu_2) = +1$$

$$\partial\Big(\underset{a}{|}\!\longrightarrow\!\underset{b}{|}\Big) = \overset{-}{\underset{a}{\bullet}} \qquad \overset{+}{\underset{b}{\bullet}} = [b]-[a]$$

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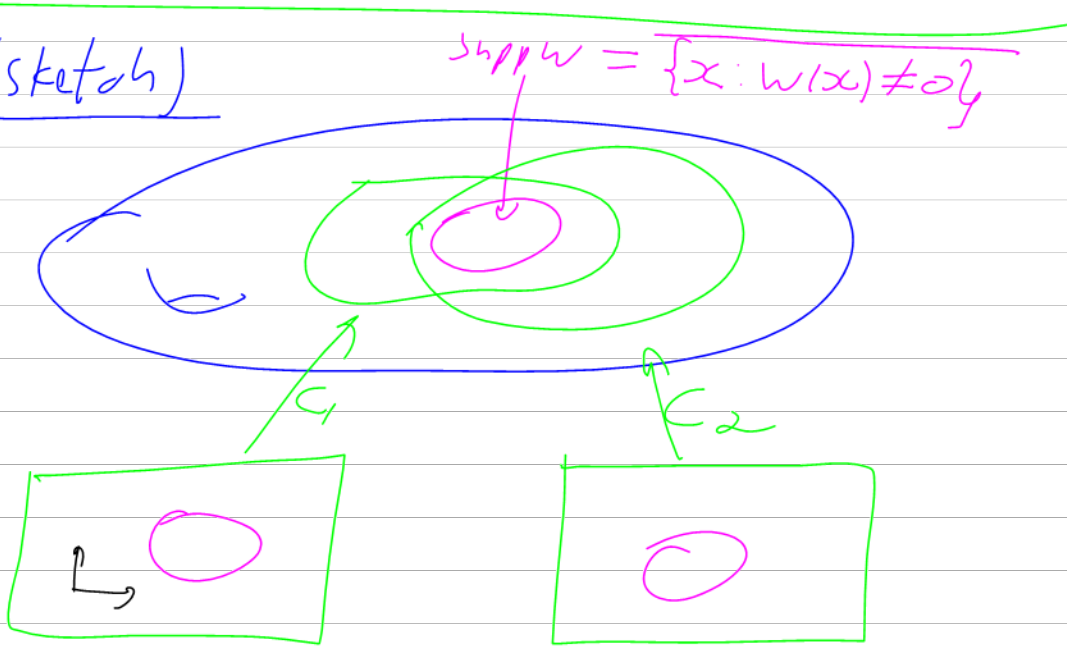
## Integration (sketch)

$M^k$ is a $k$-dim oriented mfd, $w \in \Omega^k(M)$

$$\int_M w = ?$$




**Prop** $M^k$ oriented $k$-manifold, $c_1, c_2$ are smooth injective orientation preserving cubes in $M^k$, $w \in \Omega^k(M)$ s.t. $\operatorname{supp} w \subset c_1(I^k) \cap c_2(I^k)$

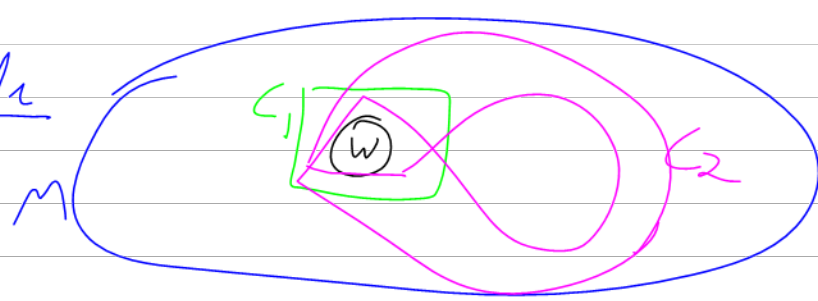
missing in Spivak 5.4.

$c_i$ is 1-1 & pushes the std orientation of $I^k$ to the orientation of $M$.

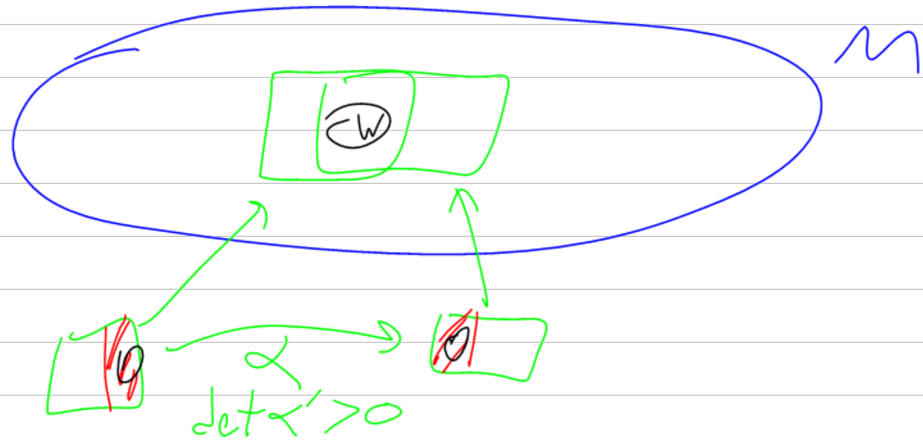
Then $$\int_{c_1} w = \int_{c_2} w$$

[Def call this common value $$\int_M w$$ (this is well defined)]

**Example**



## Proof




$P_1$ 0 1 0 1 0 0 1 1 0 1

$P_2$ 0 0 0 0 0 1 0 1

$P_3$ 0 1 1 1 0 0 1

$P_4$ 1 1 1 0 1 1 1

$$\begin{array}{ccc} 1 & \multicolumn{2}{c}{\overbrace{\hphantom{1 \quad 1}}^{y}} \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array}$$

Hour 66 MAT257 Analysis II on March 26, 2021: Integration, Stokes' Theorem. $(a+b)(a-b) \neq a^2-b^2$ ? No.
Read Along: Spivak 122-126.
Old riddle (sol'n at end). n prisoners. Each wears a tower of infinitely-many randomly-chosen b/w hats. Simultaneously each needs to point at a black hat on their head. How can they maximize the chance that they will all get it right? Can they do better than 1/2^n?

**Proposition** Let $C_i$ $(i=1,2)$ be smooth injective orientation preserving $\underbrace{k\text{-cubes}}_{\text{"good cubes"}}$ in an oriented $M^k$, and let $\omega \in \Omega^k(M)$ be s.t. $\operatorname{supp}\omega \subset C_1(I^k) \cap C_2(I^k)$ then $\int_{C_1} \omega = \int_{C_2} \omega$; **Def** Call this $\int_M \omega$

$\boxed{\text{works near } \partial M \text{ too!}}$

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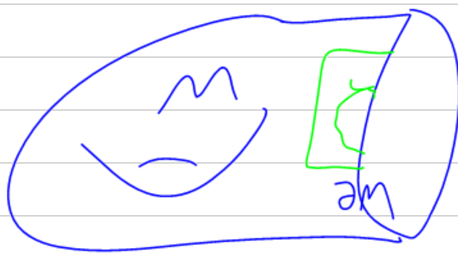
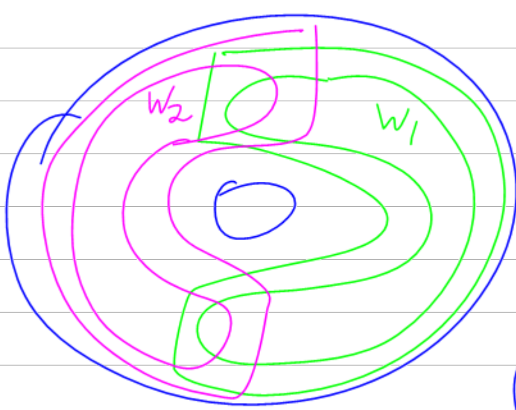


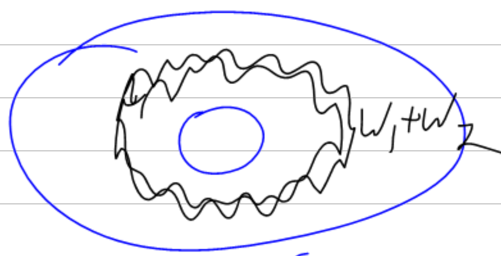

.... everything works.

**Note 1** $\int_M \omega$ is linear in $\omega$ whenever this makes sense. (when all forms are supported in the same cube)

**example**



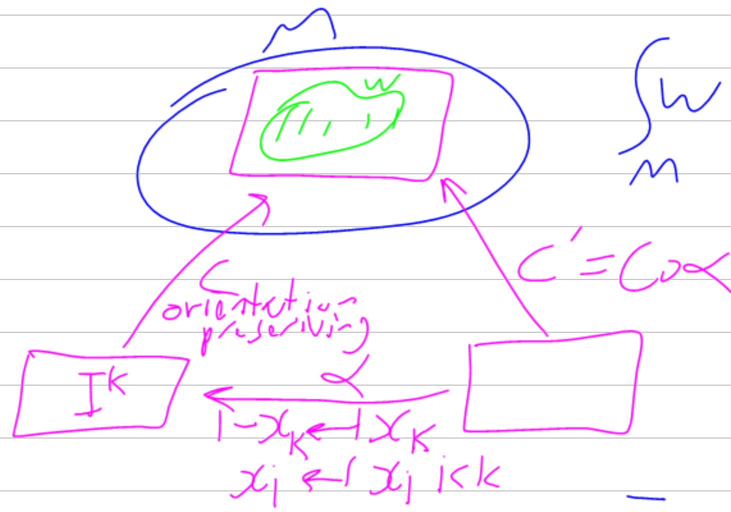

$1 < |x| < 2$




**Note 2** $\int_{-M} \omega = -\int_M \omega$

$-M$: Same as $M$, but with opposite orientation!

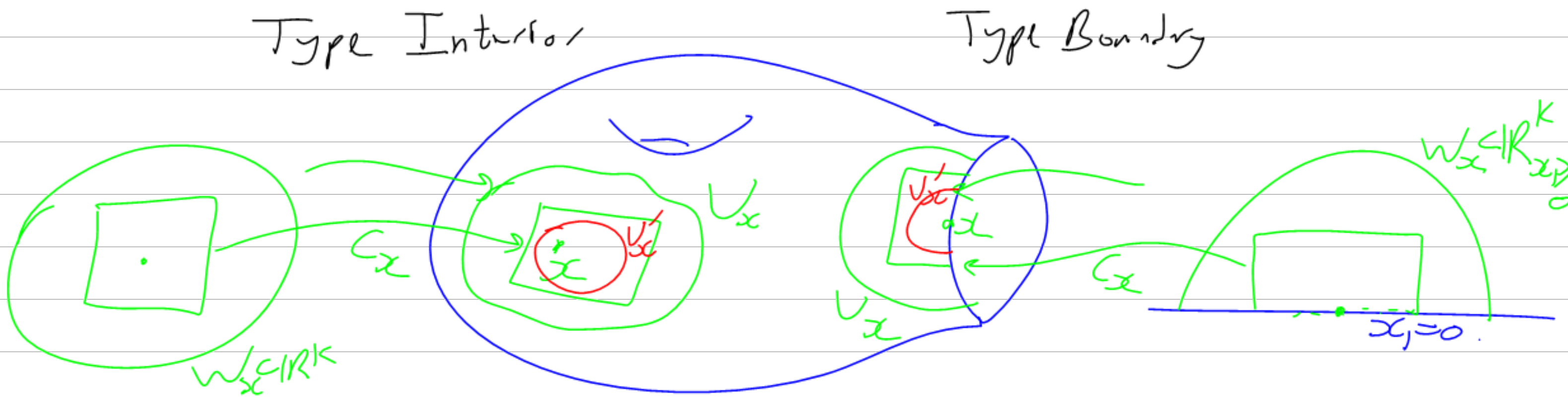

## Picture




$$\int_M \omega = \int_{I^k} c^*\omega$$

$$\int_{-M} \omega = \int_{c'} \omega = \int_{I^k} \alpha^* c^*\omega = -\int_{I^k} c^*\omega$$

To integrate a general $M$, use PO1.




We know how to integrate any form whose support $\subset U_x'$

Find a PO1 for $M$ subordinate to $V_x$ where the $V_x$'s are open in $\mathbb{R}^n$ & such that $U_x' = M \cap V_x$, call it $(\varphi_i)$

**Define** $\omega \in \Omega^k(M)$ $\displaystyle\int_M \omega = \sum_i \int_M \varphi_i \omega$ This makes sense!

Comments 1. If $M$ is cmpct, $(\varphi_i)$ can be chosen to be finite & no conv. issues. Otherwise, we say that $w$ is "integrable" if

$$\sum_i \int_{C_i} \varphi_i |w| < \infty \qquad \operatorname{supp} \varphi_i \subset \operatorname{im} C_i \qquad C_i^* w = f\, dx_1 \dots dx_k$$

$$\to \int_{I^k} \varphi_i |f|$$

and only then we use the def. above.

2. Independence of the POI is proven as before.

$$\varphi_i \quad \psi_j \quad \varphi_i \psi_j \dots$$

3. The type B pts/nbds, restricted to $\partial M$ give cubes & a POI for $\partial M$.

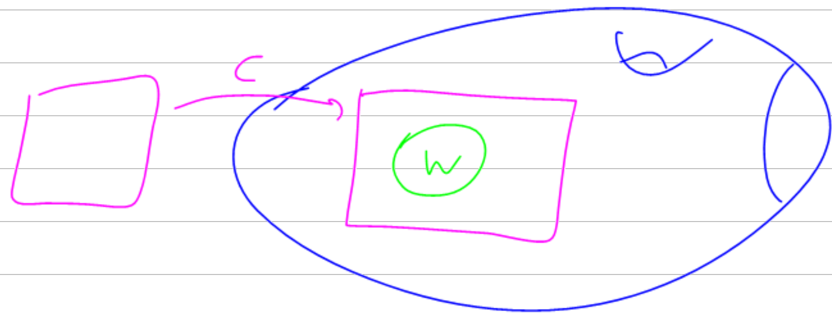
Stokes' thm If $M$ is compact & oriented, if $w \in \Omega^{k-1}(M)$, then

$$\int_M dw = \int_{\partial M} w$$

Pf Type I, then type B, then combine using a POI.

Type I Suppose $\operatorname{supp} w \subset$

$$\operatorname{int}(\operatorname{im} C) \subset \operatorname{int} M = M - \partial M$$




$$\int_M dw = \int_{I^k} c^*(dw) = \int_{I^k} d(c^* w) = \int_{\partial I^k} c^* w = \int 0 = 0$$

$$\int_{\partial M} w = 0.$$

[Not the interesting case!]