$5^5 = 3125\ abcde \subset (a+b+c+d+e)^5$

open problem!

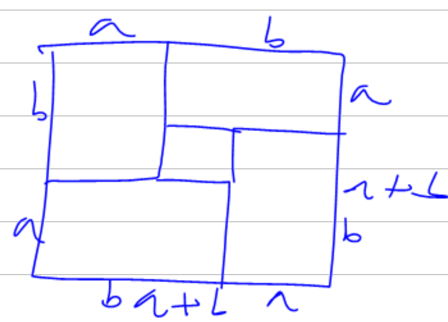
$256\ abcd$'s $\subset (a+b+c+d)^4$

easier than 3D

$27 \times abc$ boxes $\subset$ $a+b+c$ cube?

$$\sqrt[3]{abc} \le \frac{a+b+c}{3} \iff 27abc \le (a+b+c)^3$$

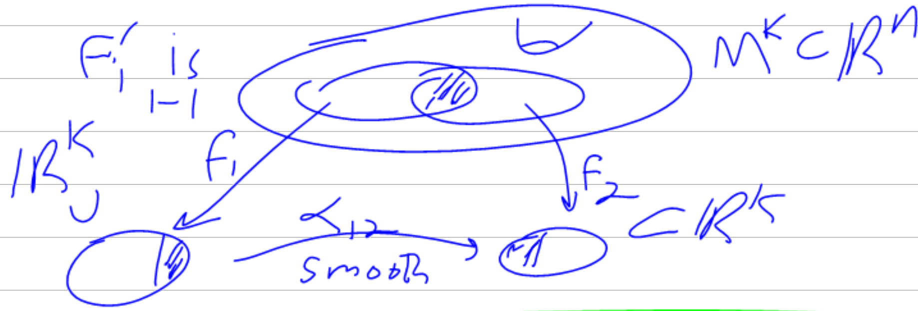
4 × 

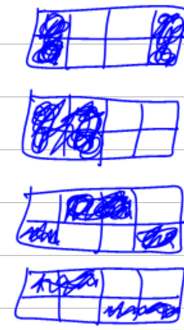
in 

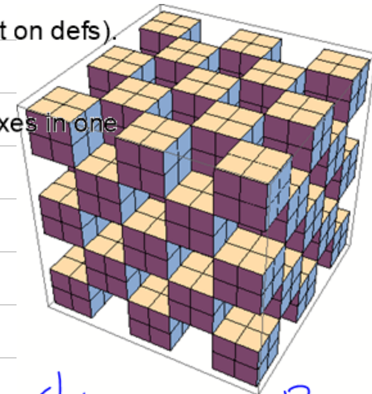
$$4ab \le (a+b)^2 \iff \sqrt{ab} \le \frac{a+b}{2}$$

"inequality of the means"

Hour 61 MAT257 Analysis II on March 15, 2021: Manifolds with boundary, forms on manifolds (weight on defs).
Read Along: Spivak 113-122.   HW17 on web by midnight!
No news yet on TT3.
Old riddle (sol'n at end): Can you fit 4 axb rectangles in one (a+b)^2 square? Can you fit 27 axbxc boxes in one (a+b+c)^3 cube? Why do I care?







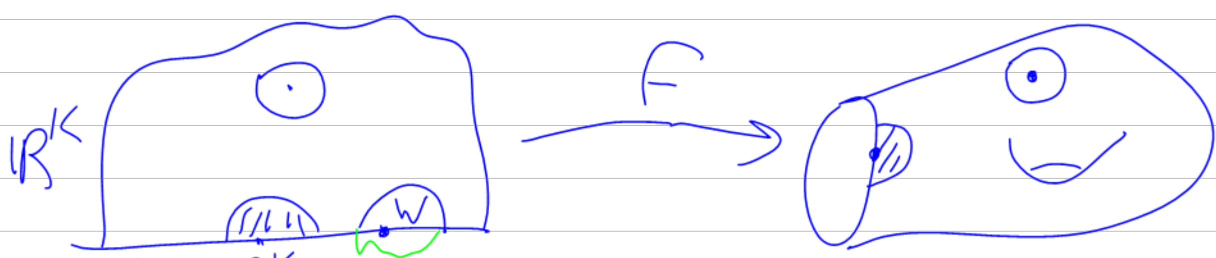
shame on me!
#black = 504

Def A subset $M \subset \mathbb{R}^n$ is a "manifold with boundary" of dim k if $\forall p \in M$ (C') holds:

(C'): There exist some open set $U \ni p$ in $\mathbb{R}^n$, some open set $W \subset \mathbb{R}^k_{x_1 \geq 0}$ & a smooth* $F: W \to U$ s.t.

1. $F(W) = M \cap U$
2. $F^{-1}: M \cap U \to W$ is cont.
3. $F'(a)$ has rank k for every $a \in W$.

* meaning F can be extended to $W' \supset W$ when W' is open in $\mathbb{R}^k$.



$H_k = \mathbb{R}^k_{x_1 \geq 0}$

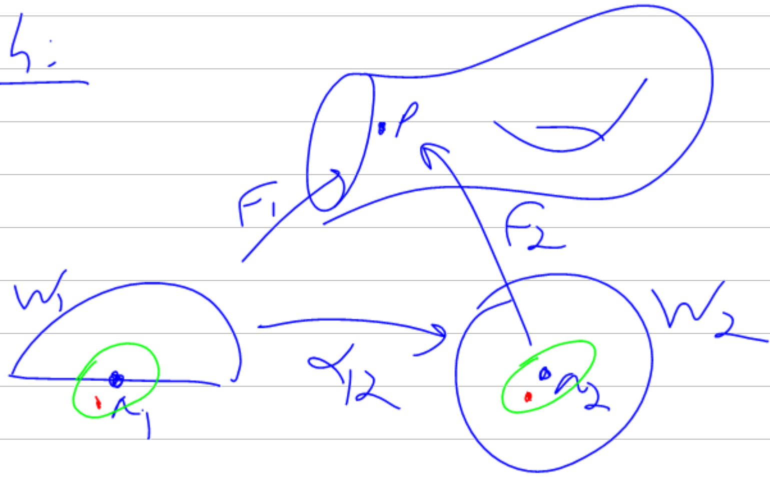
Proposition If $p \in M$ (M is a mfd w/ bndry) and $F_i: W_i \to U_i \ni p$ are coord patches (for $i=1,2$) & $f_i(x_i)=p$ then either

1. the first coord of both x_i is >0

or

2. -11- is $=0$

Sketch:



$\Longrightarrow$ If M is a mfld w/ bndry,

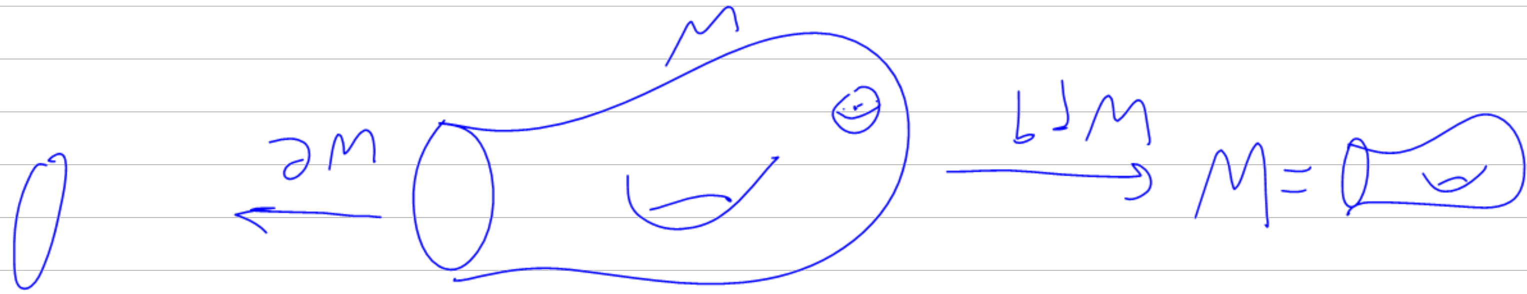
def

$$\partial M = \left\{ p \in M : \begin{array}{l} \exists \text{ coord patch} \\ f: W \subset \mathbb{R}^k_{x_1 \geq 0} \to V \ni p \\ f(x) = p \\ \text{first coord of } x = 0 \end{array} \right\}$$

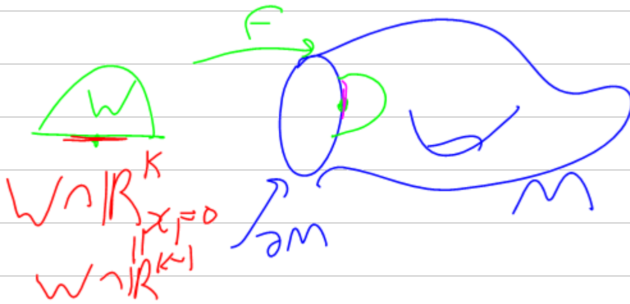
Warning $\partial M \neq \text{bd}\, M$

bndry of a mfd

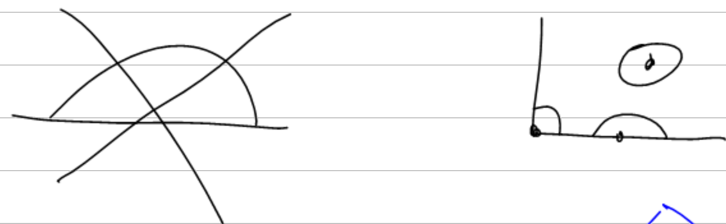
bndry of M as a subset of $\mathbb{R}^n$



Comment If M is a mfld w/ bndry then ∂M is itself a manifold of dim $k-1$ w/o bndry.



(**Aside** "manifold w/ corners"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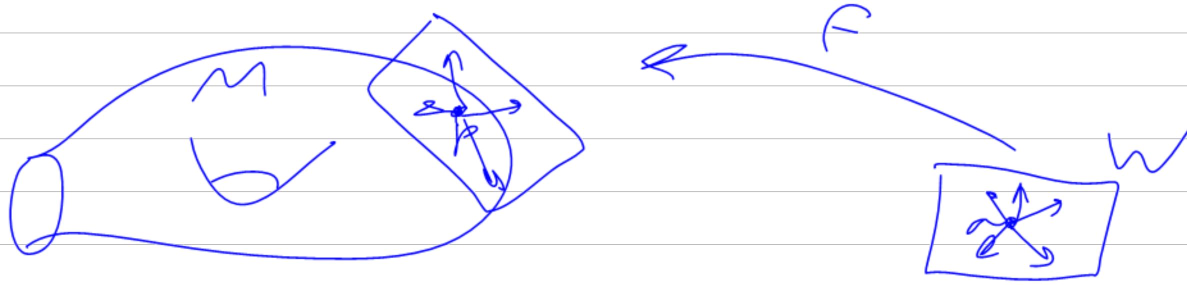
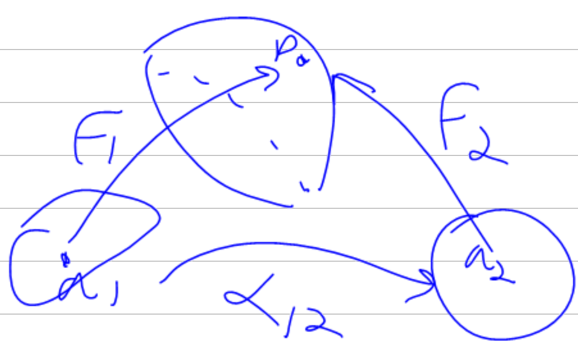


Def Given M^k, $p \in M$ where $F: W \subset \mathbb{R}^k_{x_1 \geq 0} \to M$ is a coord. patch w/ $F(a) = p$.

$$M_p = T_pM = F_* T_a \mathbb{R}^k$$

Comment 1. This is well defined:

$$F_{1*} T_{a_1} \mathbb{R}^k = F_{2*} T_{a_2} \mathbb{R}^2$$



Pf $F_1 = F_2 \circ \alpha_{12}$

So

$$\alpha_{12} = F_2^{-1} \circ F_1$$

in a nbd of a_1

$$F_{1*} T_{a_1}\mathbb{R}^k = F_{2*} \underbrace{\alpha_{12*} T_{a_1}\mathbb{R}^k}_{T_{a_2}\mathbb{R}^k} = F_{2*} T_{a_2}\mathbb{R}^k \quad \square$$

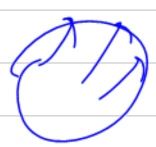
$$\dim T_pM = \dim \underbrace{F_* \underbrace{T_a\mathbb{R}^k}_{k\text{-dim}}}_{k\text{-dim}} = k.$$

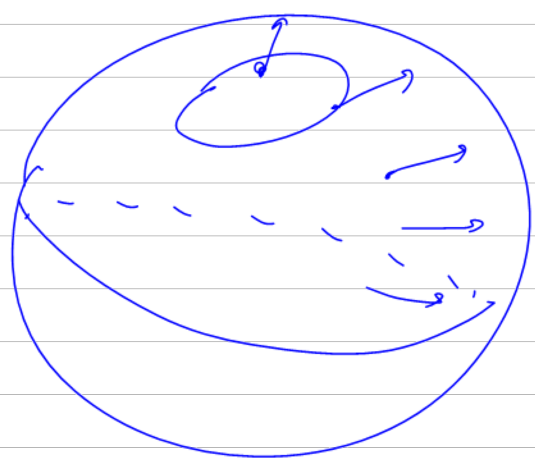
Def A vector field on a M^k is

$$F : M \longrightarrow \bigcup_{p \in M} T_pM$$

s.t.

$$F(p) \in T_pM$$





F is smooth if it is smooth as seen by any patch.

If R is a rect in $\mathbb{R}^3$

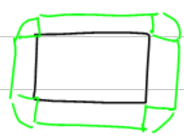
$$Vol(R_\epsilon) = Vol(R) + A(R)\cdot\epsilon + \pi S(R)\cdot\epsilon^2 + \cancel{\tfrac{4}{3}\pi\epsilon^3}$$

$$\geq$$

$$Vol(R'_\epsilon) = Vol(R') + A(R')\cdot\epsilon + \pi S(R')\epsilon^2 + \cancel{\tfrac{4}{3}\pi\epsilon^3}$$

$\Longrightarrow$ as $\epsilon \to \infty$ $\Longrightarrow$ $\pi S(R)\epsilon^2 \geq \pi S(R')\epsilon^2$ $\square$

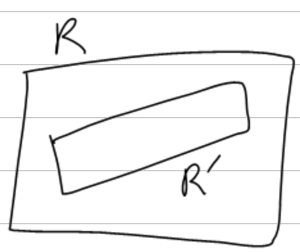
$A \subset \mathbb{R}^2$ A_ϵ: set of pts in $\mathbb{R}^2$ at most ϵ away from A

$R =$  $R_\epsilon =$ 

$$Vol(R_\epsilon) = Vol(R) + 2S(R)\epsilon + \cancel{\pi\epsilon^2}$$

$$\geq$$

$$Vol(R'_\epsilon) = Vol(R') + 2S(R')\epsilon + \cancel{\pi\epsilon^2}$$

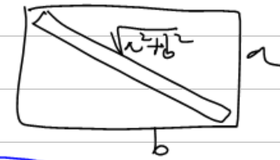


$R' \subset R \Longrightarrow R'_\epsilon \subset R_\epsilon$

$\Downarrow$?

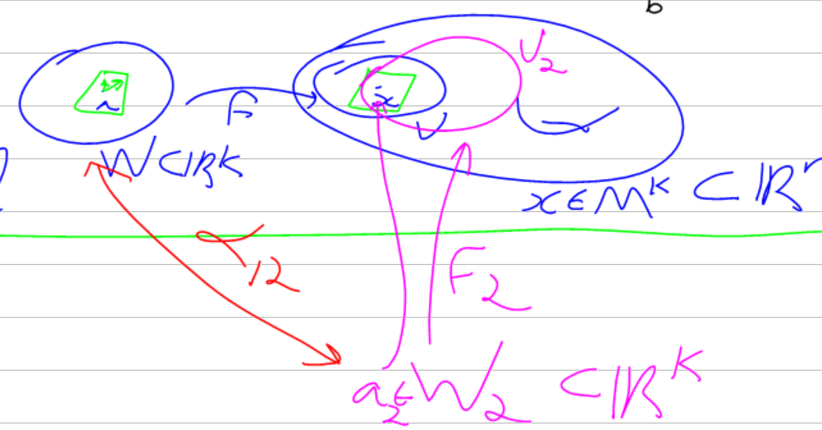
$S(R') \leq S(R)$

Hour 62 MAT257 Analysis II on March 17, 2021: Forms on manifolds, orientations.
Read Along: Spivak 113-122.
TT3 Report.
Old riddle (sol'n at end). The Moscow Subway Problem: Can you fit a box of dimensions a'xb'xc' inside a box of dimensions axbxc, if a'+b'+c'>a+b+c?



$M_x := T_x M := F_* T_a \cancel{W}\ \mathbb{R}^k$

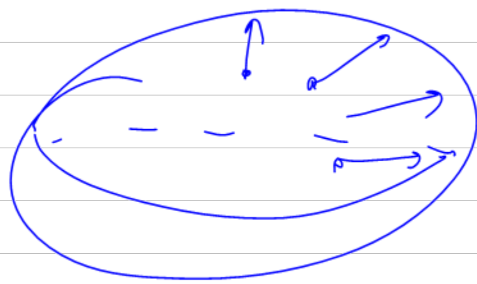
indep of the patch?



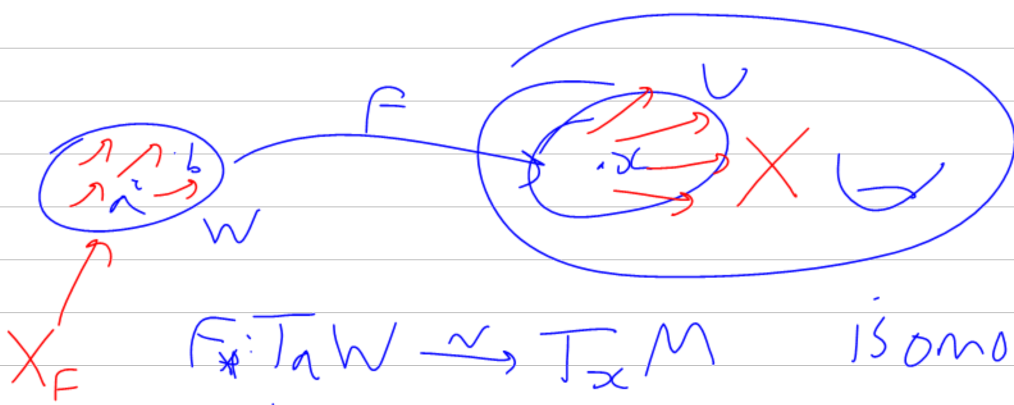
$F_* T_a W = F_{2*} T_{a_2} W_2$?

Vector fields on a mfd M

$X : M \longrightarrow \bigcup_{x \in M} T_x M \qquad X(x) \in T_x M$



X is "smooth" if it is smooth as seen in every patch.

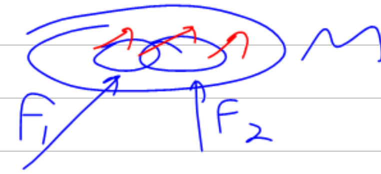


$F_* : T_a W \xrightarrow{\sim} T_x M$ isomorphism

$F^\# : T_x M \longrightarrow T_a W$ (inverse of F_*)

X_F a vector field on W by

$$X_F(b) = F^{\#}(X(F(b)))$$



Def X is smooth if for every coord chart F, X_F is smooth.

Lemma (skipped) It is enough to check smoothness on a union of coord. charts that cover M.

$$X+Y, \quad aX$$

make sense on a manifold & obey the same rules

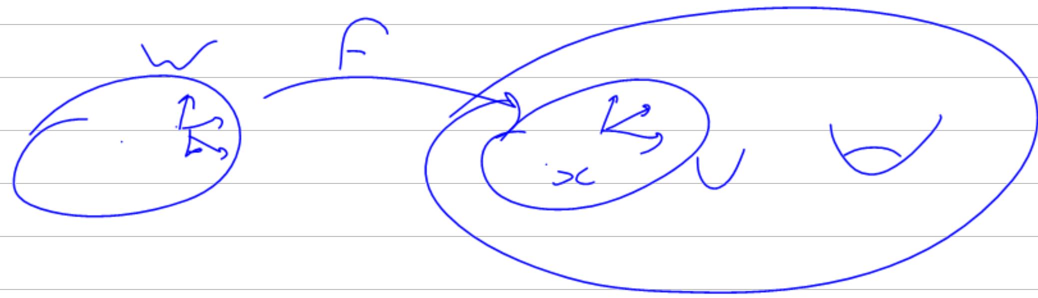
Def A p-form on $M^k \subset \mathbb{R}^n$ $(p \leq k \leq n)$ is a machine

$$w: M \longrightarrow \bigcup_{x \in M} \Lambda^p(T_x M)$$

s.t.

$$w(x) \in \Lambda^p(T_x M)$$

In other words w takes p tangent vectors to M, all based at the same point x, an spits out a number, s.t. it is multi-linear & alternating.



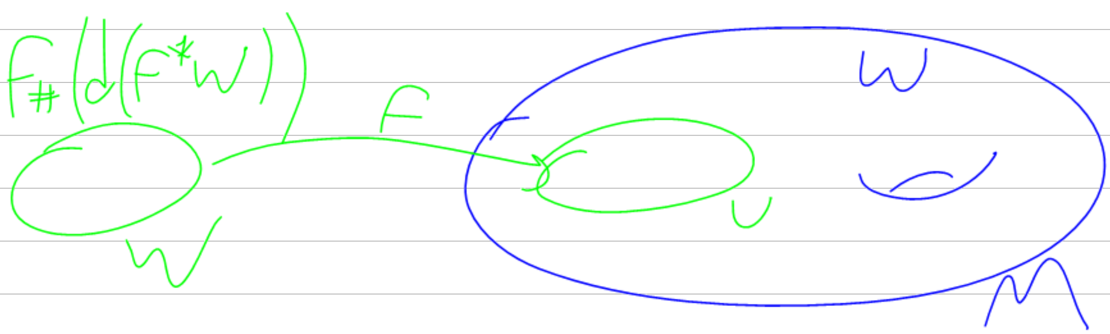
if ω is a p-form on M, $F^*\omega$ is a p-form on W.

F^* identifies p-forms on $U \subset M$ w/ p-forms on W.

We say that a p-form ω on M is smooth if for every patch, $F^*\omega$ is smooth.

$\Omega^p(M)$: all smooth p-forms on M

$\Omega^p(M)$ has $+$, $a\cdot$, $\wedge$, $d: \Omega^p(M) \to \Omega^{p+1}(M)$

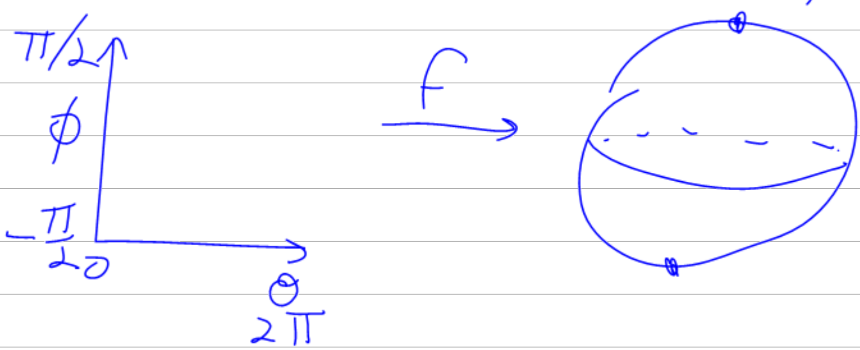


All properties of $+$, a, $\wedge$, d still hold.

$$d(\omega \wedge \eta) = (d\omega)\wedge \eta + (-1)^{\deg \omega}\, \omega \wedge d\eta$$

Take $M = S^2 = \{a \in \mathbb{R}^3 : \|a\|^2 = 1\}$

$a = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad x^2+y^2+z^2 = 1$



$$f(\theta,\phi) = \begin{pmatrix} \cos\phi \cdot \cos\theta \\ \cos\phi \,\sin\theta \\ \sin\phi \end{pmatrix}$$

$$\Omega^0(M) \ni x,\ y,\ z,\ x^2+y^2$$

$$x^2+y^2+z^2 = 1$$

$$\Omega^1(M) \ni dx,\ dy,\ dz,\ d(x^2+y^2) = 2x\,dx + 2y\,dy$$

$$2x\,dx + 2y\,dy + 2z\,dz = d1 = 0$$

$x\,dx + y\,dy = 0$?

$5 \cdot 0 = 7 \cdot 0$

$5 = 7$?

To check if it is 0, pull it back via a chart

$$\cos\phi\cos\theta\, d(\cos\phi\cos\theta) + \cos\phi\sin\theta\, d(\cos\phi\sin\theta)$$

$$= \ldots\ldots \neq 0.$$

$$x\,dx + y\,dy = -z\,dz \xrightarrow{F^*} -\sin\phi\, d\sin\phi$$

$$= -\sin\phi\cos\phi\, d\phi \neq 0$$

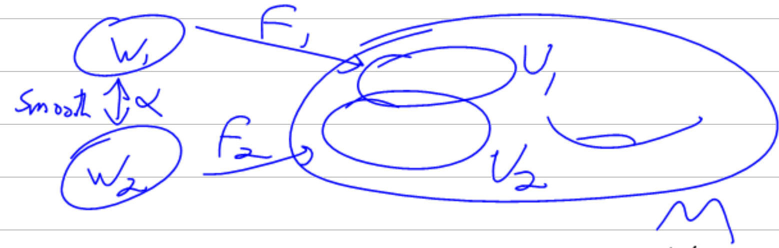
$$\Omega^2(S^2) \ni x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy = \eta$$

$$d\eta = \underset{\text{on } S^2}{\overset{3\,dx\wedge dy\wedge dz \;\neq_{\text{in } \mathbb{R}^3} 0}{0}}$$

Hour 63 MAT257 Analysis II on March 19, 2021: Orientations.
Read Along: Spivak 113-122.
Old riddle (sol'n at end). On any pair of ~~potatoes~~ apples, can you draw a pair of 3D congruent curves?

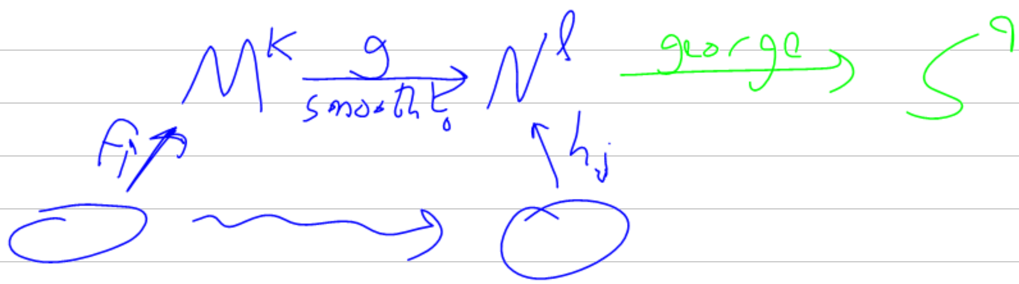
Keyword: ghosting!

Work patch by patch, use the smooth α to show that patches agree: T_xM, $\Omega^1(M)$, $+$, $\wedge$, d.



Sometimes hard to tell if $\omega_1 = \omega_2$. $\omega = x\,dx + y\,dy + z\,dz \in \Omega^1(S^2)$

Comments 1

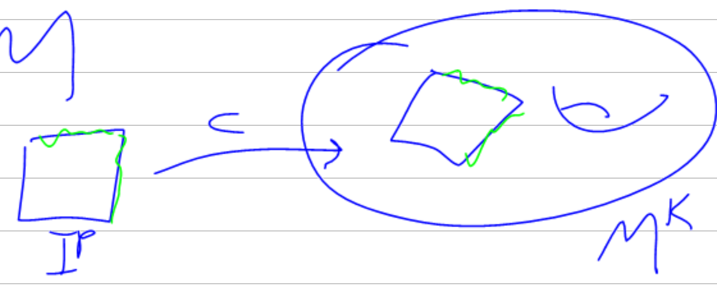


Def g is smooth if for every two coord patches f for M & h for N, $h^{-1} \circ g \circ f$ is smooth where defined.

Can use smooth maps to push/pull things between M & N. All $\mathbb{R}^n$ rules apply.

Comment 2 Cubes in M

$C_p(M)$, ∂

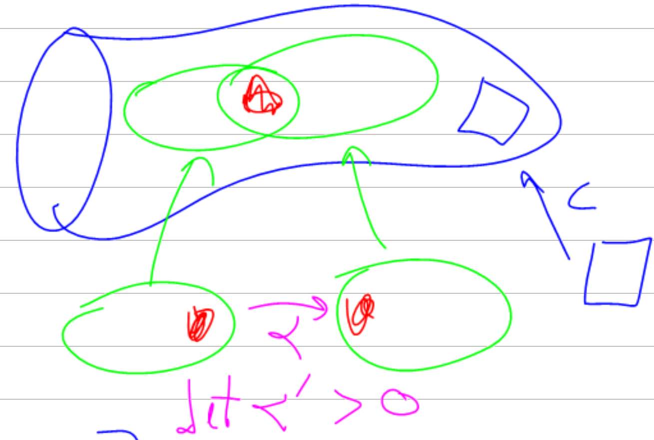


$\partial^2 = 0$, $\int_{\text{chains}}$ form

Stokes' thm still holds for chains in M.

This isn't the Stokes thm we want!

Q Given M, can we always find a system of coord patches for M which covers it all ["an atlas"] s.t. all transition functions will have $\det \alpha' > 0$?

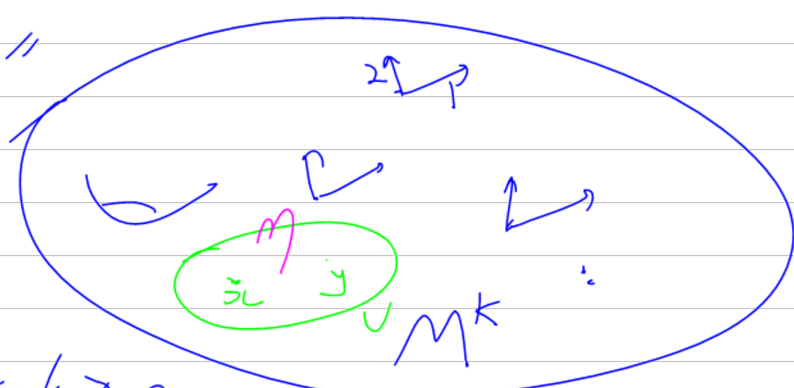


Reminder V n-dim vect. space V^n

Orientation for V:

$$\frac{\text{ordered basis}}{\text{pos det changes of basis}} \sim \frac{\eta \in \Lambda^n(V)}{\text{mult. by pos. scalars}}$$

Def An "orientation" on a mfld M is a $\boxed{\text{cont. varying}}$ choice of an orientation for T_xM for every $x \in M$.

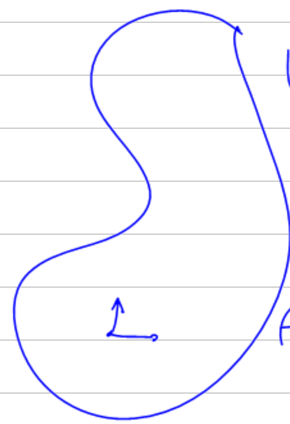


Cont. varying: For every $x \in M$, we can

Find a nbd U & a smooth non-vanishing k-form η on U s.t. $\eta(y)$ is the orientation of T_yM for every $y \in U$.

$\Longleftrightarrow$ For every $x \in M$ we can find a nbd U & k smooth vector fields $X_1 \dots X_k$ on U s.t. $(X_1(y), \dots, X_k(y))$ represents the orientation of T_yM, for every $y \in U$.

Examples



$U^n \subset \mathbb{R}^n$

open set in $\mathbb{R}^n$

For T_xU choose $(e_1, \dots, e_n)$

std basis of $\mathbb{R}^n \simeq T_xU$

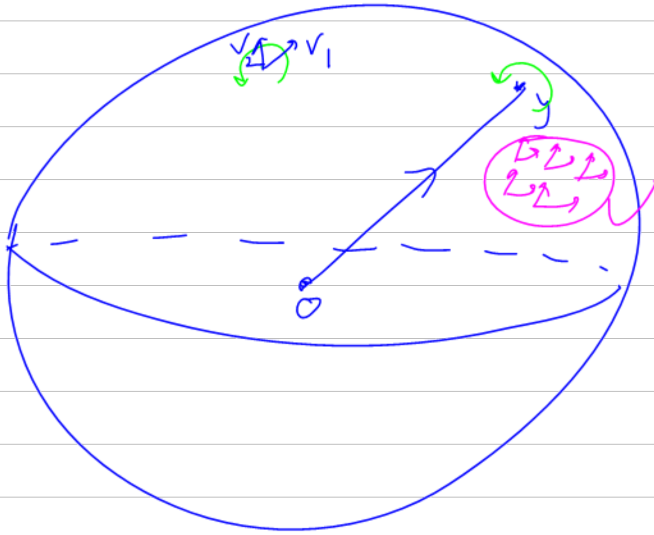
Example

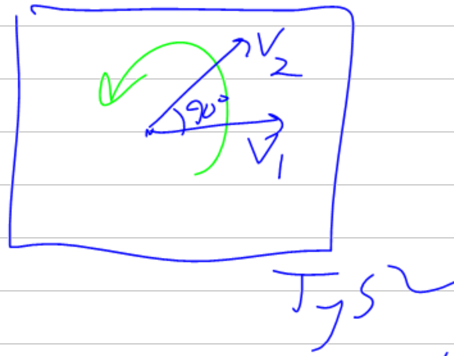
$S^2 =$

"cannot comb a sphere"

every cont vector field on S^2 has zeros.

$\Rightarrow$ no globally defined X_1 & X_2 that make a basis of T_yS^2 for every $y \in S^2$.



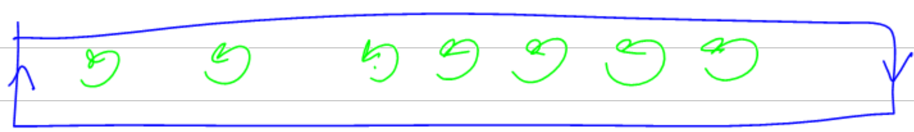


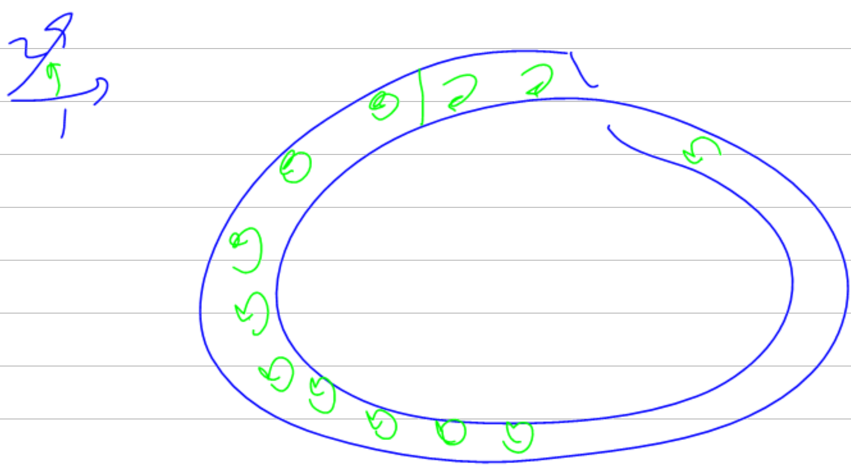
Found an orientation for S^2!

If M has an orientation we say that it is "orientable".

If M comes w/ an orientation we say it is "oriented".

Möbius band:





is not orientable!