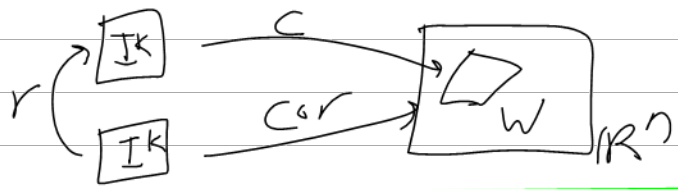
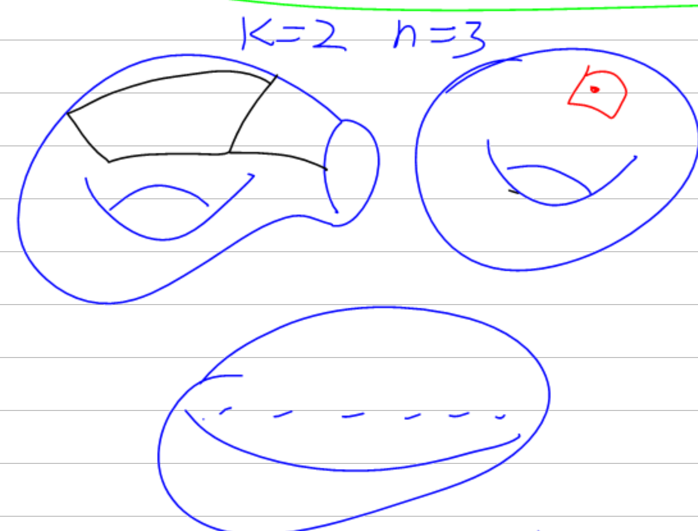
Hour 58 MAT257 Analysis II on March 8, 2021: Manifolds.
Read Along: Spivak 109-115.
TT3 is tomorrow! Info and prep questions at http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT3-Info.pdf

Exercise Given $c: I^k \to A \subset \mathbb{R}^n$, $\omega \in \Omega^k(A)$ and smooth $r: I^k \to I^k$ 1-1, onto, w/ $\det r' > 0$, $\int_c \omega = \int_{c\circ r} \omega$



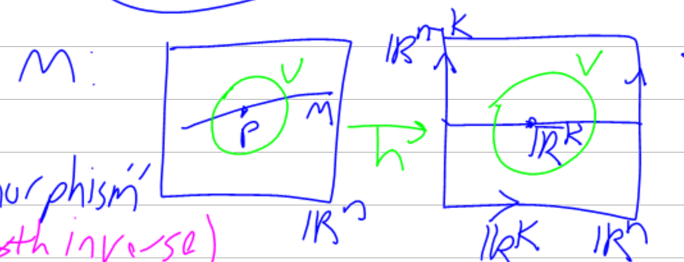
These are "k-dim manifolds in $\mathbb{R}^n$"



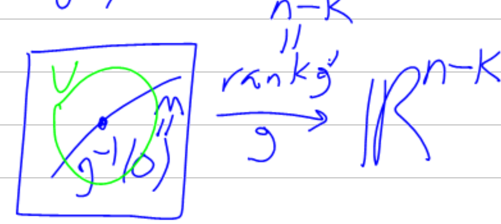
Thm Given $k \le n$, $M \subset \mathbb{R}^n$, $p \in M$, TFAE:

(M) $\exists\, U \ni p$ open, $\exists\, V$ open in $\mathbb{R}^n_{\text{other}}$ & $h: U \to V$ "a diffeomorphism" (smooth, w/ smooth inverse)

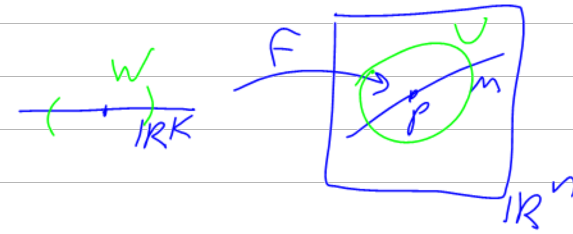
s.t. $h(U \cap M) = V \cap (\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\})$



(Z) $\exists$ open $U \ni p$ & smooth $g: U \to \mathbb{R}^{n-k}$ s.t. $U \cap M = U \cap g^{-1}(0)$ & $\operatorname{rank}(g') = n-k$



(C) $\exists$ open $U \ni p$, open $W \subset \mathbb{R}^k$ & smooth $\underbrace{F}_{1-1}: W \to \mathbb{R}^n$ s.t.



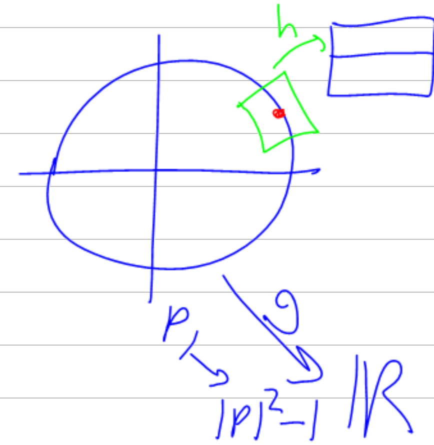
① $F(W) = M \cap U$ ② $F^{-1}: M \cap U \to W$ is cont.

③ $\forall a \in W$ rank $F'(a) = k$

Def Given $k \le n$, $M \subset \mathbb{R}^n$ is a k-dim manifold if $\forall p \in M$ any one of the conditions above hold.

k-manifold M^k

Examples 1. $S^1 \subset \mathbb{R}^2$, indeed



(Z):
$$S^1 = \{p \in \mathbb{R}^2 : |p| = 1\}$$
$$= \{p \in \mathbb{R}^2 : |p|^2 - 1 = 0\}$$
$$= g^{-1}(0)$$

$g(x,y) = x^2 + y^2 - 1$

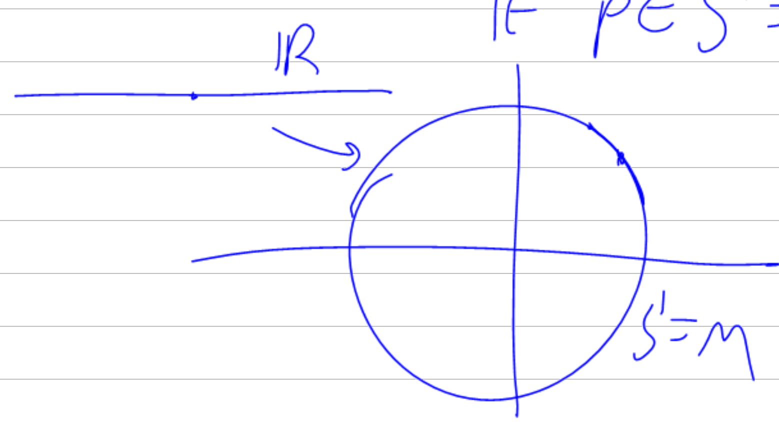
$$g' = \left(\frac{\partial g}{\partial x} \;\; \frac{\partial g}{\partial y}\right) = (2x \;\; 2y) \ne 0$$

if $p \in S^1 = M$

(C) Use

$F : \mathbb{R}_t \longrightarrow \mathbb{R}^2_{x,y}$

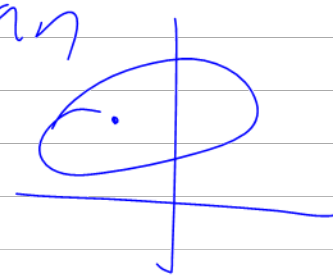
$F(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$



$$F' = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \ne 0 \quad \operatorname{rank} F' = 1$$

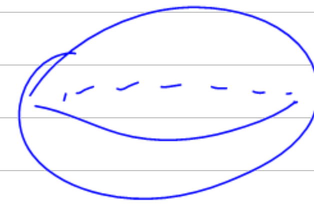
0. Any open set in $\mathbb{R}^n$ is an n-manifold in $\mathbb{R}^n$

Any finite set in $\mathbb{R}^n$ is a 0-manifold in $\mathbb{R}^n$



2. $S^n = \{ p \in \mathbb{R}^{n+1} : |p| = 1 \}$

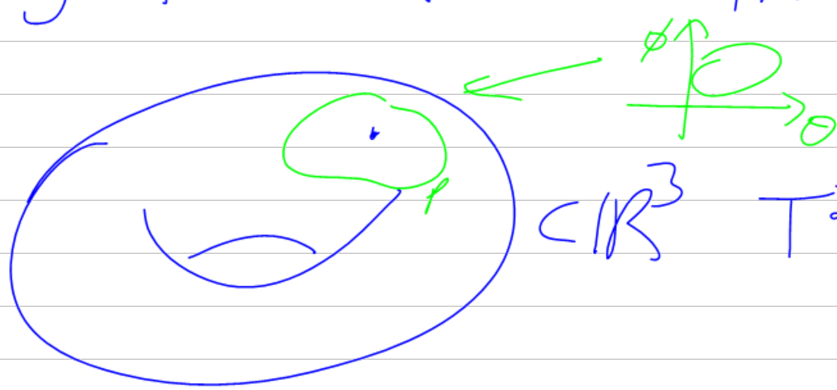
$S^2 \subset \mathbb{R}^3$



a manifold as

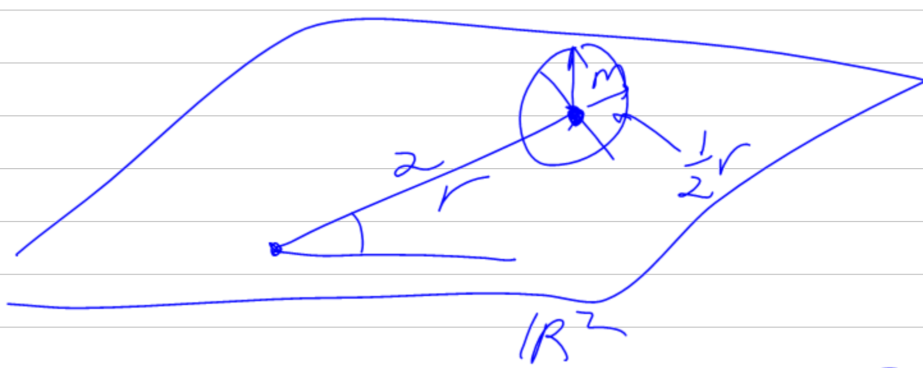
$S^n = g^{-1}(0)$ where

$g : \mathbb{R}^{n+1} \to \mathbb{R} \quad p \mapsto |p|^2 - 1$

3. $T^2 =$ 

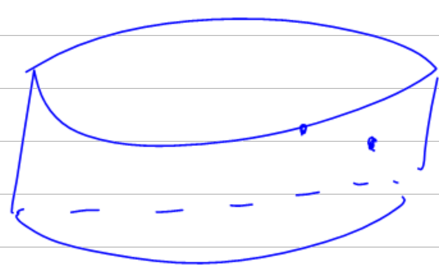
$\subset \mathbb{R}^3$

$$T^2 = \left\{ \begin{pmatrix} (2+\cos\phi)\cos\theta \\ (2+\cos\phi)\sin\theta \\ \sin\phi \end{pmatrix} \right\}$$



$$\begin{pmatrix} r \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}r \\ 0 \end{pmatrix} \cos\phi + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sin\phi$$

$$r = 2 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



$$\int_a^b f(x)\,\underline{\underline{dx}}$$

$$\int_{[a,b]} f\,\underline{\underline{dx}}$$

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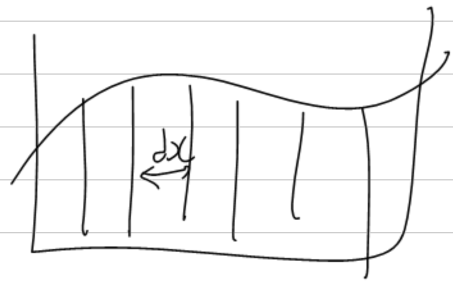


$dx\wedge dy$

$dy\wedge dx = -dx\wedge dy$

$$dx_I = dx_{i_1}\wedge\overbrace{(dx_{i_2}\wedge\cdots\wedge dx_{i_k})}^{k-1}$$

$$\underbrace{\omega}_{1}\wedge\underbrace{\eta}_{k-1}$$

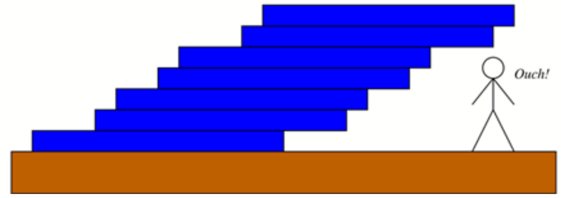
$$\underbrace{d\omega}_{k} = \sum \underbrace{dx_i}_{1}\wedge\underbrace{\frac{\partial\omega}{\partial x_i}}_{k}$$

Hour 59 MAT257 Analysis II on March 10, 2021: Manifolds.
Read Along: Spivak 109-115.
HW16 is on the web!
Our final assessment: Tuesday April 20 9-12 (morning TO time).

Within 10 minutes of the start of the test yesterday, the questions were posted on a question-sharing web site. Somebody thinks they are smart! We'll work to prove them wrong.

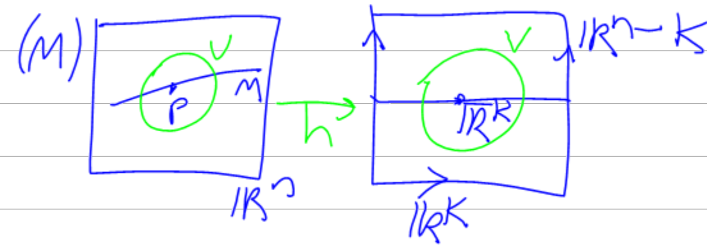
There was an issue with Q4; it will be managed after I know how many students were affected.

How far sideways can you reach by stacking up n identical blue domino pieces, before your tower will lean over and fall? What if n goes to infinity?
(no glue is allowed, and as shown, the stacking isn't necessarily "even")

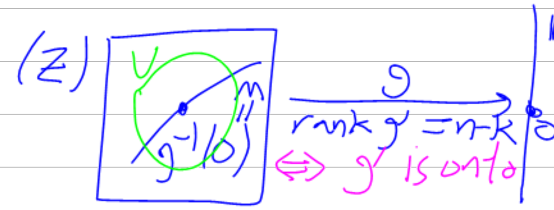


Thm Given $k\le n$, $M\subset\mathbb{R}^n$, $p\in M$, TFAE:

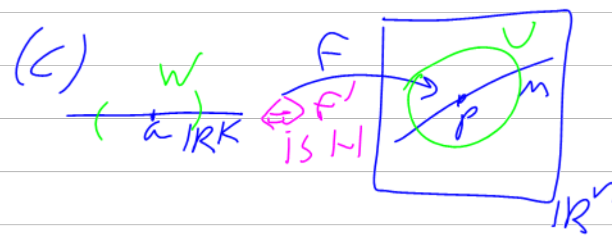
(M) $\exists$ open $U\ni p$, open $V\subset\mathbb{R}^n_{\text{other}}$ & a diffeomorphism $h:U\to V$ s.t. $h(U\cap M)=V\cap(\mathbb{R}^k\times\{0_{\mathbb{R}^{n-k}}\})$.



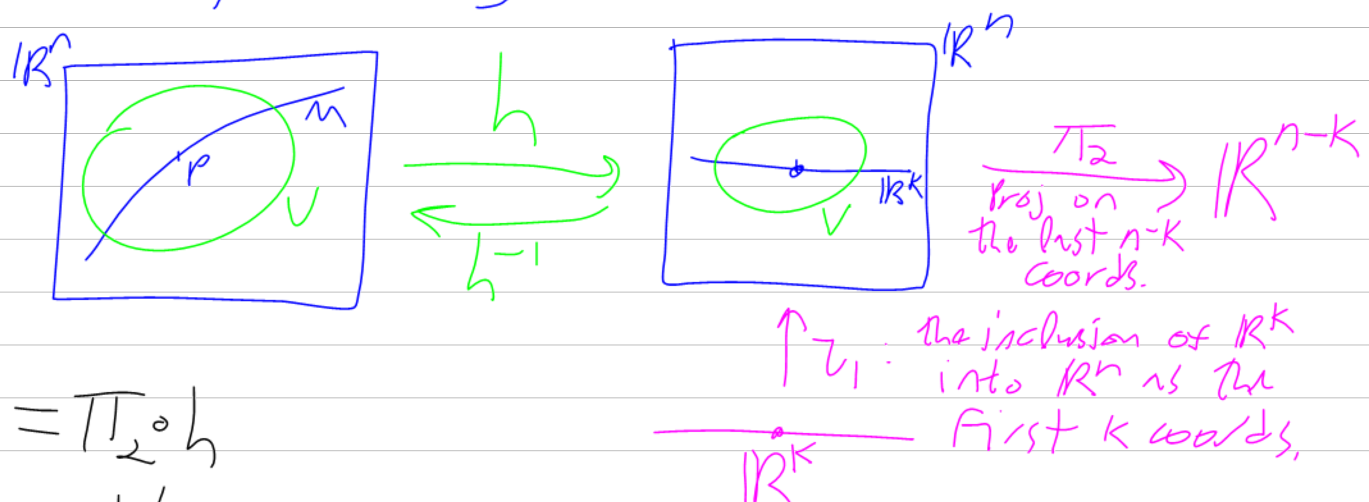
(Z) $\exists$ open $U\ni p$ & smooth $g:U\to\mathbb{R}^{n-k}$ s.t. $U\cap M=U\cap g^{-1}(0)$ & $\operatorname{rank}(g')=n-k$



(C) $\exists$ open $U\ni p$, open $W\subset\mathbb{R}^k$ & smooth 1-1 $f:W\to\mathbb{R}^n$ s.t. ① $f(W)=M\cap U$ ② $f^{-1}:M\cap U\to W$ is cont. ③ $\forall a\in W$, $\operatorname{rank} f'(a)=k$.



PF M$\Rightarrow$C,Z easy. Indeed

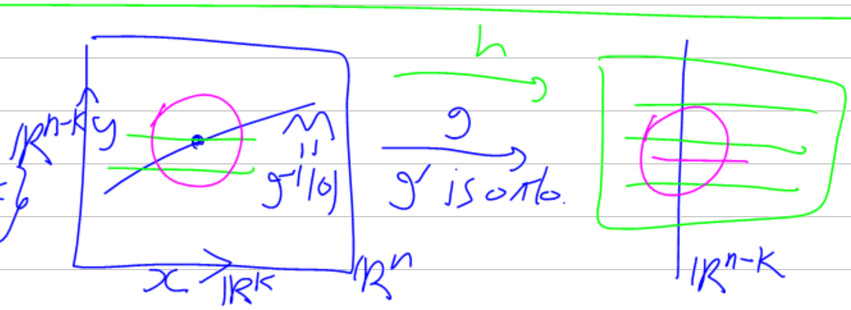


(Z) Set $g=\pi_2\circ h$

(C) set $f=h^{-1}\circ\iota_1$

(Z)$\longrightarrow$(M)

$\mathbb{R}^n=\{(x,y)\in\mathbb{R}^k\times\mathbb{R}^{n-k}\}$



$$g' = \begin{array}{c} \quad K \qquad n-K \\ \left(\frac{\partial g}{\partial x} \,\middle|\, \frac{\partial g}{\partial y}\right)\Big\}{n-K} \\ \underbrace{\qquad\qquad}_{n} \end{array}$$

wlog $\frac{\partial g}{\partial y}$ these cols are lin-indep meaning that $\frac{\partial g}{\partial y}$ is invertible

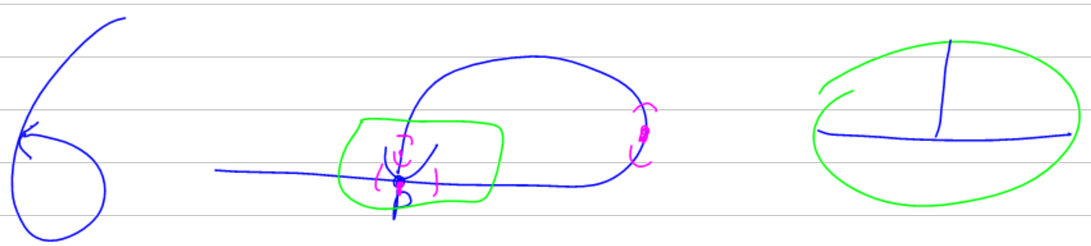
set $h(x,y) = (x, g(x,y))$

$$h' = \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} I & 0 \\ * & \frac{\partial g}{\partial y} \end{pmatrix}$$ is invertible

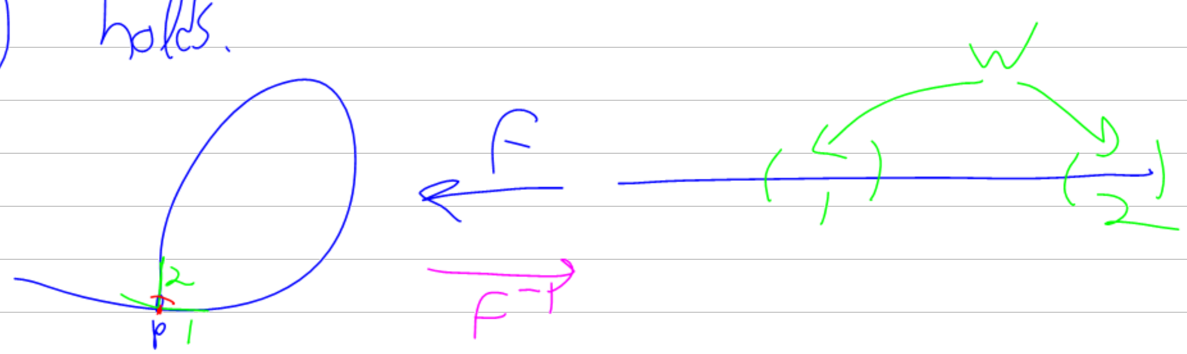
invertible

hence by the inverse function thm, h is invertible near p.

Aside In (C) cond ② is necessary $6 \in \mathbb{R}^2$:



(C)\(2) holds.

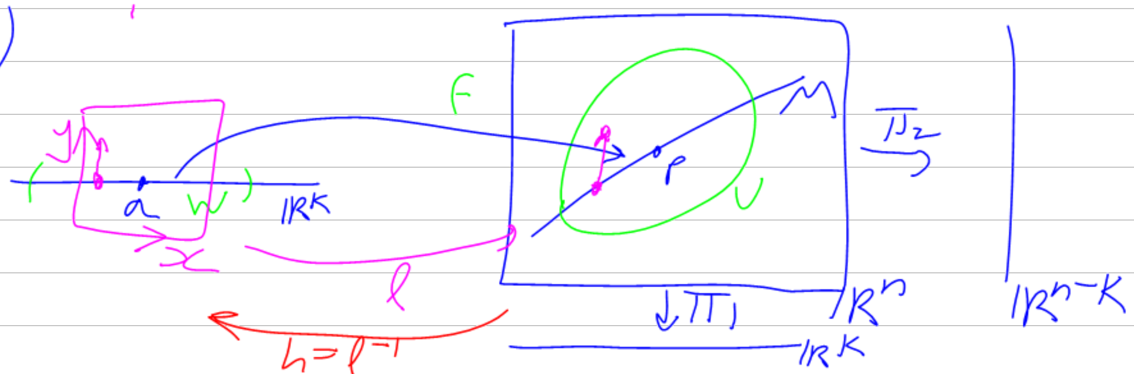


Pf of (C) $\Rightarrow$ (M)

1 $F(W) = M \cap U$

2 $F^{-1}: M \cap U \to W$ is cont

③ rank $F'(a) = K$ on all of W

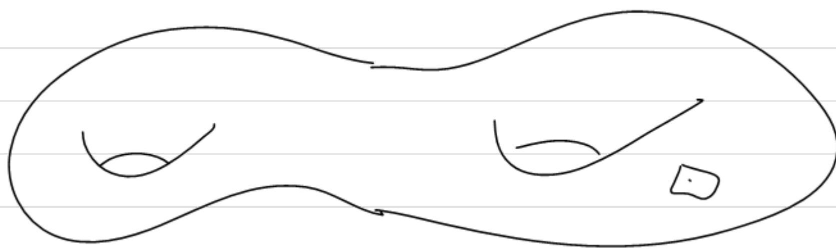


WLOG, $\operatorname{rank}(\pi_1 \circ f) = k$

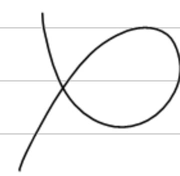
Define $\ell(x,y) = f(x) + \begin{pmatrix} 0 \\ y \end{pmatrix}$

$$\ell : \mathbb{R}^k \times \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^n \qquad \ell' = \begin{pmatrix} \frac{\partial f}{\partial x} & 0 \\ \frac{\partial f}{\partial y} & I \end{pmatrix}$$

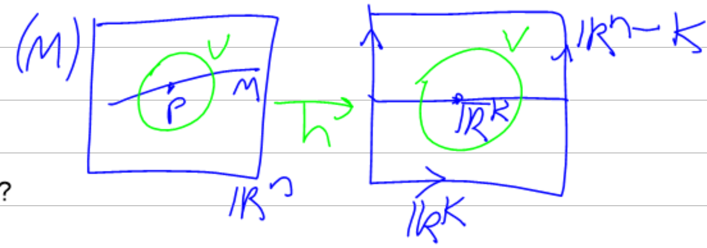
$$\mathbb{R}^n \longrightarrow$$

 $\subset \mathbb{R}^3$ 

$\sigma : (0,1) \xrightarrow[1-1]{} \mathbb{R}^2$

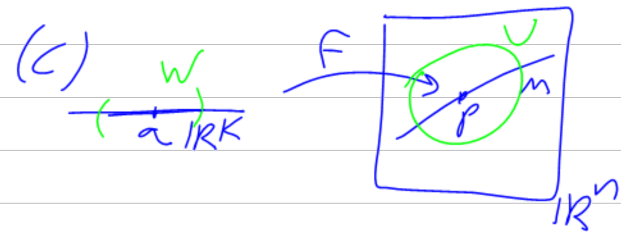


Hour 60 MAT257 Analysis II on March 12, 2021: Manifolds, Coord patches, manifolds with boundary.
Read Along: Spivak 109-115.
No news yet on TT3.
Our final assessment: Tuesday April 20 9-12 (morning TO time).
Old riddle (sol'n at end): Can you fit 125 4x2x1 boxes in one 10x10x10 cube?



$\frac{1}{2}$Thm Given $k\leq n$, $M\subset\mathbb{R}^n$, $p\in M$, $(C)\Rightarrow(M)$

(M) $\exists$ open $U\ni p$, open $V\subset\mathbb{R}^{\text{other}}$ & a diffeomorphism $h: U\to V$ s.t. $h(U\cap M)=V\cap(\mathbb{R}^k\times\{0_{\mathbb{R}^{n-k}}\})$.



(C) $\exists$ open $U\ni p$, open $W\subset\mathbb{R}^k$ & smooth 1-1 $f: W\to\mathbb{R}^n$ s.t.
① $f(W)=M\cap U$ ② $f^{-1}: M\cap U\to W$ is cont. ③ $\forall a\in W$, $\operatorname{rank} f'(a)=k$.

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wlog
$$\begin{tikzcd} W\subset\mathbb{R}^k \arrow[r, "f"] & \mathbb{R}^k_x\times\mathbb{R}^{n-k}_y=\mathbb{R}^n \arrow[d, "\pi_1"'] \arrow[d, shift left=6, "\pi_2"] \\ & \mathbb{R}^k \qquad \mathbb{R}^{n-k} \end{tikzcd}$$

$f'=\underbrace{\left(\equiv\right)}_{k}\,n$

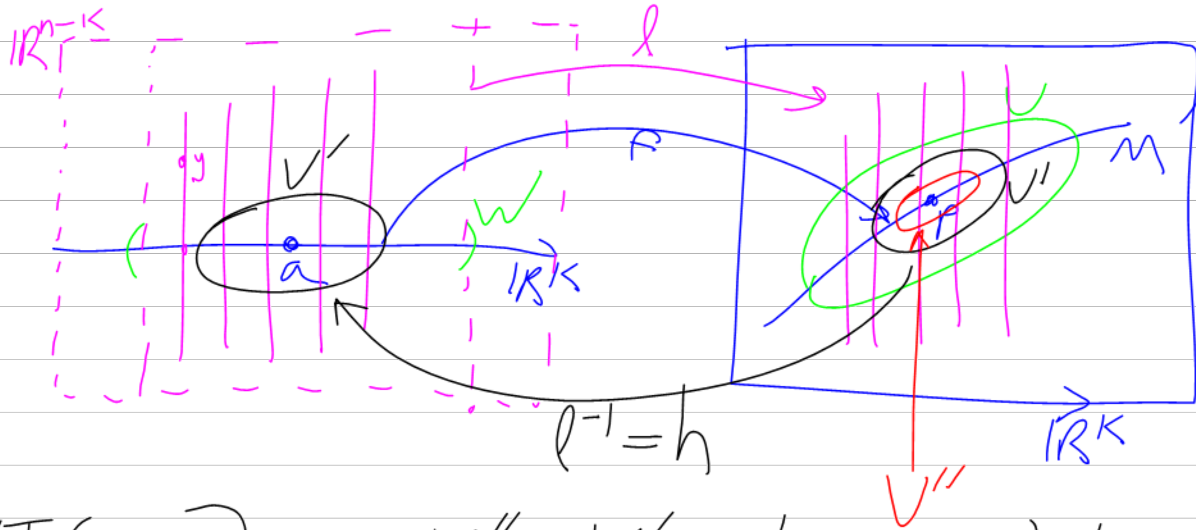
$(\pi_1\circ f)'$ is invertible.

$$\ell: W_x\times\mathbb{R}^{n-k}_y\longrightarrow\mathbb{R}^k\times\mathbb{R}^{n-k}=\mathbb{R}^n$$

by
$$\ell:(x,y)\longmapsto f(x)+\begin{pmatrix}0\\y\end{pmatrix}\in\mathbb{R}^n$$

$$\ell'=\begin{array}{c|cc} & k & n-k\\ \hline k & \multicolumn{2}{c}{\multirow{2}{*}{$\left(\begin{array}{c|c}\frac{\partial(\pi_1\circ f)}{\partial x} & 0\\ \hline \frac{\partial(\pi_2\circ f)}{\partial x} & I\end{array}\right)$}}\\ n-k & & \end{array}$$ invertible at a.

$\Rightarrow\exists$ V' & U' s.t. $\ell|_{V'}: V'\longrightarrow U'$
is invertible where $V'\subset W\times\mathbb{R}^{n-k}$
& $U'\subset U$



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N.T.S. $\exists$ open $U'' \subset U'$ s.t. $V'' = h(U'')$, then

$$h(U'' \cap M) = V'' \cap \mathbb{R}^k$$

$$\Updownarrow$$

$$U'' \cap M = \ell(V'' \cap \mathbb{R}^k)$$

NTS $\quad F(V'' \cap \mathbb{R}^k) = U'' \cap M$

Recall $F^{-1} : M \cap U \longrightarrow W$ is cont.

meaning $F : W \longrightarrow M \cap U$ is "open"

carries open sets to sets of the form $\tilde{U} \cap (M \cap U)$ where $\tilde{U} \subset \mathbb{R}^n$ is open

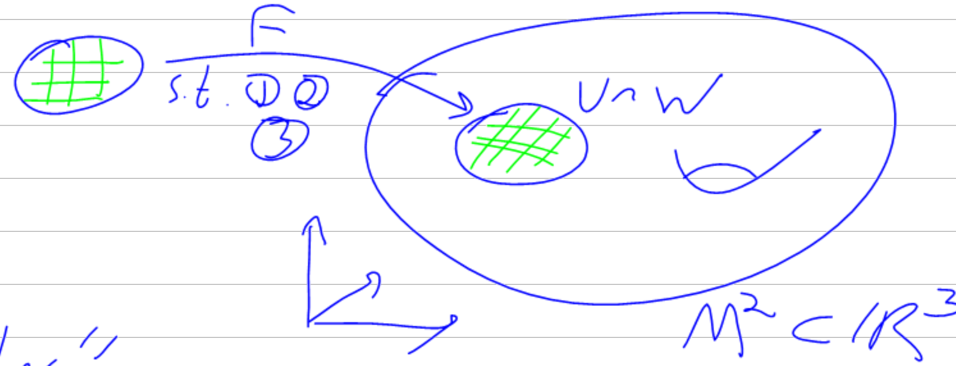
$\Rightarrow F(V' \cap \mathbb{R}^k) = \tilde{U} \cap (M \cap U)$

where $\tilde{U} \subset \mathbb{R}^n$ is open

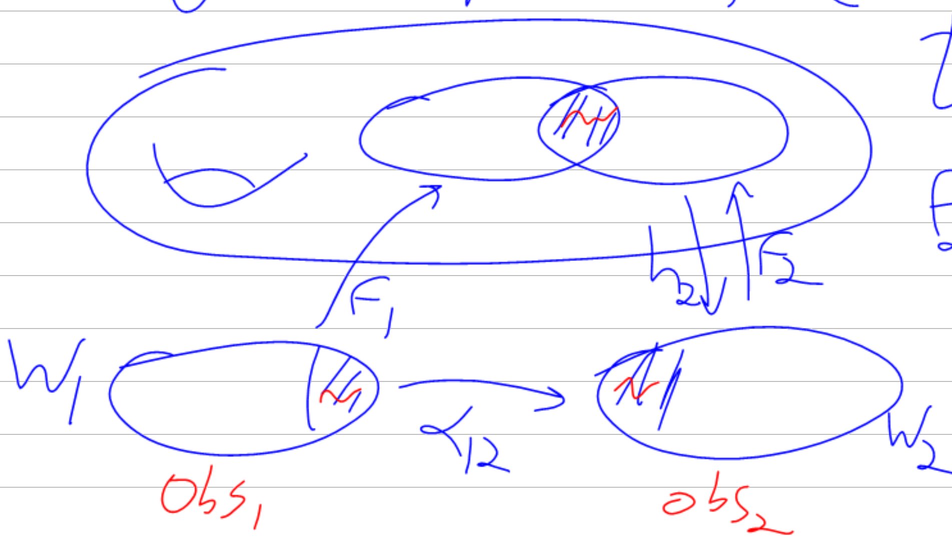
$$= M \cap \underbrace{(U \cap \tilde{U})}_{U''} = M \cap U''$$

what we wanted, w/ $V'' = V'$.

Def If $M^k \subset \mathbb{R}^n$, the f's of (C) are called "coordinate patches".



Corollary Suppose f_1, f_2 are coord. patches



then $\alpha_{12} = f_2^{-1} \circ f_1 : f_1^{-1}(f_2(W_2)) \longrightarrow f_2^{-1}(f_1(W_1))$ is smooth w/ a smooth inverse.

$f_2^{-1} \circ f_1 = \pi_1 \circ h_2 \circ f_1$ so it is smooth.