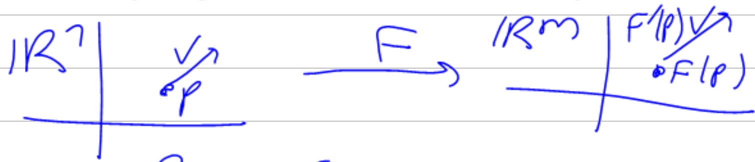
$$
\begin{tikzcd}
& H & & H & & H \\
A\text{-}U_1 \arrow[ur, ">"] \arrow[dr, "<"'] \arrow[rr, "="] & & AU_1 - U_2U_3 \arrow[ur, ">"] \arrow[dr, "<"'] \arrow[rr, "="] & & AU_1U_2U_3 - U_4U_5 - U_6U_7 \arrow[ur, ">"] \arrow[dr, "<"'] \arrow[r, "="] & {} \\
& L & & L & & L
\end{tikzcd}
$$

MAT257 Analysis II on February 8, 2021: Vector fields and differential forms.
Read Along: Spivak 86-92.
HW14 will be on the web by midnight. It will be due on the next teaching Monday.
Riddled before: Is there a continuous surjection f:[0,1]->[0,1] which is constant on a set of intervals whose lengths sum to 1?
Riddled before: A unit cube in R^3, the area of its projection on any plane is equal to the length of its projection on a perpendicular line to that plane.
Just learned from Tanya Khovanova, https://blog.tanyakhovanova.com/2021/02/the-anniversary-coin/: Eight out of sixteen coins are heavier than the rest and weigh 11 grams each. The other eight coins weigh 10 grams each. We do not know which coin is which, but one coin is conspicuously marked as an "Anniversary" coin. Can you figure out whether the Anniversary coin is heavier or lighter using a balance scale at most three times?



Reminder. Given $p \in \mathbb{R}^n$,

$\mathbb{R}^n_p \sim T_p\mathbb{R}^n := \{(p, v) : v \in \mathbb{R}^n\} = \{v_p\}$ Pushes!

$F : \mathbb{R}^n \to \bigcup_{p \in \mathbb{R}^n} T_p\mathbb{R}^n$ s.t. $F(p) \in T_pV$ is a "Vector field"

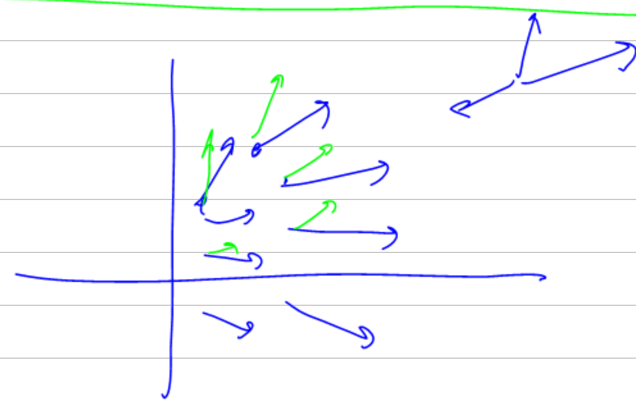
Can add, scale, inner-multiply, but not push or pull.

$T_p\mathbb{R}^n = \langle (p, e_i) \rangle$

$\in$

$$F(p) = \sum_{i=1}^{n} F^i(p)(p, e_i)$$

where $F^i : \mathbb{R}^n \to \mathbb{R}$



$\{\text{Vector fields}\} \longleftrightarrow \left\{\begin{array}{l}\text{sequences of}\\ n \text{ function}\end{array}\right\} \longleftrightarrow \{F : \mathbb{R}^n \circlearrowleft\}$

Def A V.F. F is cont. / diffable / cont. diffable / smooth / C^r if $\forall i$ F^i is cont. / diff / ⋮ / smooth / ⋮

Tangent vectors & directional derivatives

Suppose $\xi = (p, v) \in T_p\mathbb{R}^n$ suppose $f : \mathbb{R}^n \to \mathbb{R}$ (defined near p) is diffable.

$$D_\xi f \overset{1}{=} f'(p)\cdot v$$

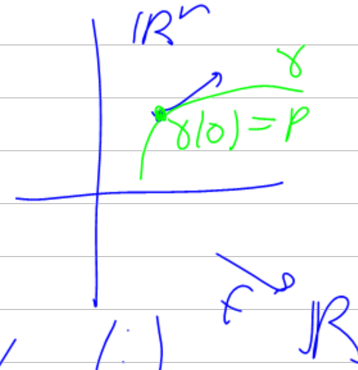
$$\overset{2}{=} (f\circ\gamma)'(0)$$

Pick diff'ble $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ defined near 0 s.t.

$$\gamma(0) = p$$

$$\gamma'(0)\cdot e_1 = v \quad \begin{pmatrix} \vdots \end{pmatrix}$$

$$\begin{pmatrix} \vdots \end{pmatrix} (1) \quad \text{basis of } \mathbb{R}^1$$



Claim $1=2$.

Pf use chain rule:

$$2 = (f\circ\gamma)'(0) = f'(\gamma(0))\cdot\gamma'(0) = f'(p)\cdot v = 1.$$

Properties of $D_\xi f$

1. "bilinear" linear in ξ & in f

$$D_\xi(af+bg) = a D_\xi f + b D_\xi g$$

$$D_{a\xi+b\zeta} f = a D_\xi f + b D_\zeta f$$

2. "Local" If $f_1 = f_2$ near p (meaning, in a small ball containing p)

then $$D_\xi f_1 = D_\xi f_2$$

$$\boxed{(fg)' = fg' + f'g}$$

3. Leibnitz' rule:

Pf: Exercise

$$D_\xi(f\cdot g) = f(p)\cdot D_\xi g + g(p)\cdot D_\xi f$$

Another word on push/pull

$$\mathbb{R} \xrightarrow{\gamma} \mathbb{R}^n \xrightarrow{F} \mathbb{R}^m \xrightarrow{g} \mathbb{R}$$

$\xi = (p, V)$ $\qquad$ $(F(p), F'(p)\cdot V) = F_*\xi$

$\gamma(0) = p$

$\gamma'(0) = V$

$\dot\gamma(0) = \xi$ where $\dot\gamma(t) = (\gamma(t), \gamma'(t)e_1) = \gamma_*(t, e_1)$

<u>claim</u> 1. $D_{F_*\xi}\, g = D_\xi (F^* g)$

2. $(F_* \dot\gamma)(0) = F_*(\dot\gamma(0))$

$\boxed{\text{The other half is yours to complete}}$ (So does 2)

<u>$\frac{1}{2}$pf</u> 1 follows from chain rule.

$$D_{F_*\xi}\, g = g'(F(p))\, F_*(\xi) =$$

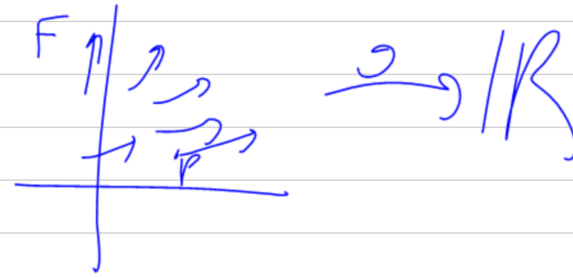
if $\xi = (p, V)$, $F_*\xi = (F(p), F'(p)\cdot V)$

$$= g'(F(p)) \cdot F'(p) \cdot V$$

$$= (g \circ F)'(p) \cdot V = D_{(p,V)}(g \circ F) = D_\xi (F^* g) \quad \square$$

<u>Minor comment</u> Similarly, vector fields differentiate functions resulting in fnctns.

If F is a v.f. on $\mathbb{R}^n$, & $g : \mathbb{R}^n \to \mathbb{R}$ diff'rble then $(D_F g)(p) = D_{F(p)} g$



Q/Riddle If F & G are V.F., so is

$$[F, G] = D_F \circ D_G - D_F \circ D_G : \{\text{fncts}\} \circlearrowleft$$

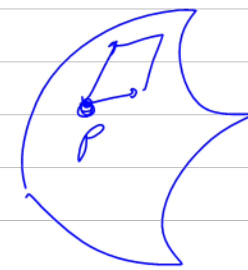
$$D_F, D_G : \{\text{fncts}\} \longrightarrow \{\text{fncts}\}$$

Some sources use the notation: $D_F g = F \cdot g$

MAT257 Analysis II on February 10, 2021: Differential forms and d.
Read Along: Spivak 86-92.
Reminder - TA Office Hours: Sebastian Monday 11-12, Shuyang Wednesday 3-4; a great resource!
Learned from Tanya Khovanova, https://blog.tanyakhovanova.com/2021/02/the-anniversary-coin/: Eight out of sixteen coins are heavier than the rest and weigh 11 grams each. The other eight coins weigh 10 grams each. We do not know which coin is which, but one coin is conspicuously marked as an "Anniversary" coin. Can you figure out whether the Anniversary coin is heavier or lighter using a balance scale at most three times?

Def A k-form, or a differential form of degree k, is an assignment:

$$\int_C d\omega = \int_{\partial C} \omega$$



$$\lambda : \mathbb{R}^n \longrightarrow \bigcup_p \Lambda^k(T_p\mathbb{R}^n)$$

s.t. $\lambda(p) \in \Lambda^k(T_p\mathbb{R}^n)$ $\quad \xi_i = (p, v_i)$

$$(\xi_1 \dots \xi_k) \longmapsto \lambda(\xi_1, \dots, \xi_k) \in \mathbb{R}$$

provided ξ_i belong to the same tangent space.

$$\lambda(p) = \sum_{I \in \Omega_n^k} \underbrace{\lambda_I(p)}_{\in\,\mathbb{R}} \cdot w_I$$

where $w_I \in \Lambda^k(T_p\mathbb{R}^n)$ that is $I = (i_1, \dots i_k)$

$\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$

where $\varphi_j(p, v) = v_j$ ← jth entry of v.

$$\lambda_I : \mathbb{R}^n \longrightarrow \mathbb{R}$$

λ is called cont. / diff-able / smooth when $\forall I$ λ_I is cont. / diff-able / smooth.

Def $\Omega^k(\mathbb{R}^n) = \left\{ \begin{array}{c} \text{all smooth} \\ \text{k-forms on } \mathbb{R}^n \end{array} \right\}$

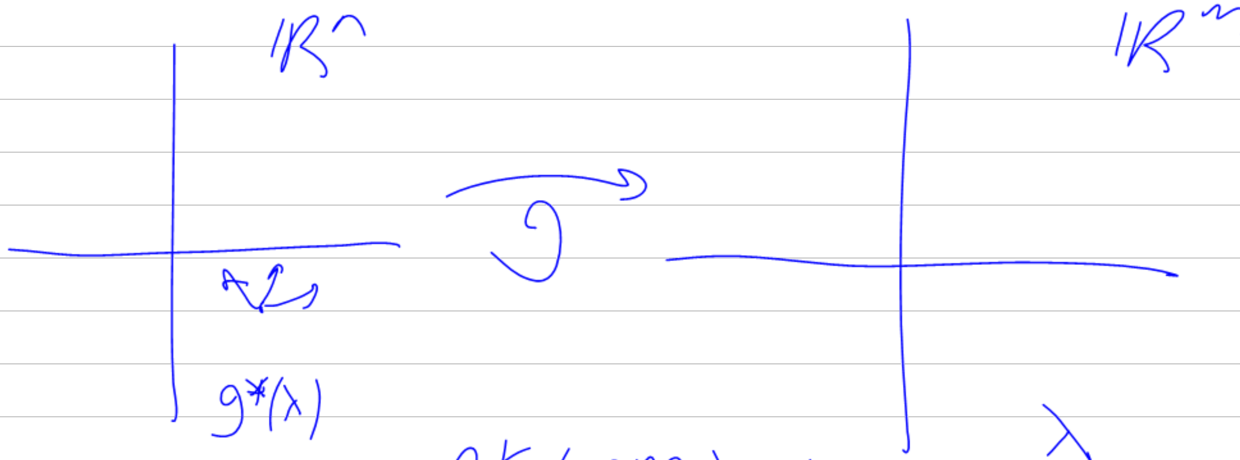
Let $\lambda \in \Omega^k(\mathbb{R}^n)$ be diff'able (a slight misuse of language)

Def/claims/no proofs:

Ω^k has $+$, $\alpha\cdot$, hence a v.s.

has $\wedge : \Omega^k \times \Omega^\ell \longrightarrow \Omega^{k+\ell}(\mathbb{R}^n)$

associative, super commutative...



<u>Def</u> Given $\lambda \in \Omega^k(\mathbb{R}^m)$ define

$$g^*\lambda \in \Omega^k(\mathbb{R}^n) \text{ by}$$

$$(g^*\lambda)(\xi_1 \ldots \xi_k) = \lambda(g_*\xi_1 \ldots g_*\xi_k)$$

<u>claims</u> This is compatible with $+$, $\alpha\cdot$, $\wedge$:

$$g^*(\lambda \wedge \eta) = g^*(\lambda) \wedge g^*(\eta)$$

$\frac{1}{2}$ <u>Pf</u>

$$g^*(\overset{\Omega^k}{\lambda} \wedge \overset{\Omega^\ell}{\eta})(\xi_1 \ldots \xi_{k+\ell}) = (\lambda \wedge \eta)(g_*\xi_1 \ldots g_*\xi_\ell)$$

$$= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma \lambda(g_*\xi_{\sigma 1}, \dots) \, \eta(\dots, g_*\xi_{\sigma(k+\ell)})$$

now compute rhs in similar way and get the same.

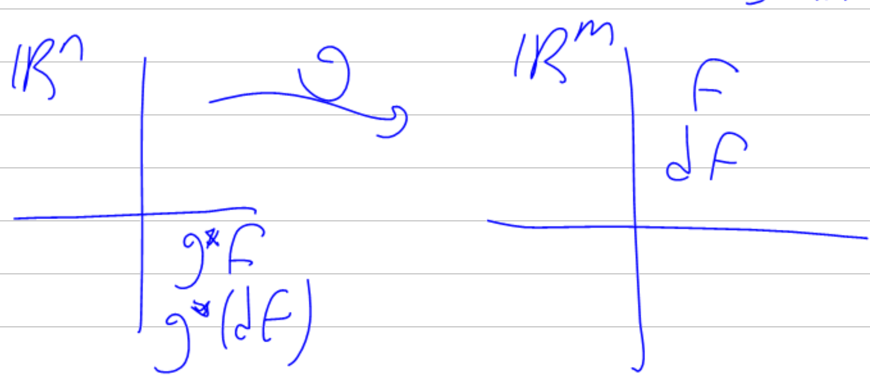
Def $\quad d : \Omega^0(\mathbb{R}^n) \longrightarrow \Omega^1(\mathbb{R}^n)$

"the differential" "the exterior derivative".

$$\Omega^0(\mathbb{R}^n) = \left\{ \begin{matrix} \text{smooth} \\ \text{functions} \\ \text{on } \mathbb{R}^n \end{matrix} \right\} = \left\{ \lambda_{\{\}} \cdot w_{\{\}} \right\} \ni f$$

$$\Omega^1(\mathbb{R}^n) = \left\{ \begin{matrix} \text{assignments that assign a} \\ \text{number to any single} \\ \text{tangent vector to } \mathbb{R}^n \end{matrix} \right\}$$

by $\quad (df)(\xi) = D_\xi f = f'(p) \cdot v \qquad \xi \in T_p\mathbb{R}^n$, $\quad \xi = (p, v)$, $\quad p \in \mathbb{R}^n$.



Claim $\quad d(g^*f) = g^*(df) \quad$ in $\Omega^1(\mathbb{R}^n)$

Indeed, let $\xi \in T_p(\mathbb{R}^n)$

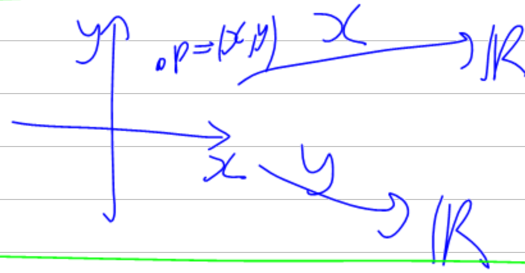
$$(d(g^*f))(\xi) = D_\xi(g^*f) =$$

$$(g^*(df))(\xi) = (df)(g_*\xi) = D_{g_*\xi} f = \text{—}$$ done on Monday

Example $\quad \mathbb{R}^n_{x_1 \dots x_n} \qquad x_i : \mathbb{R}^n \to \mathbb{R}$

$x' = (1 \quad 0)$

$y' = (0 \quad 1)$



$\xi = (p, v)$

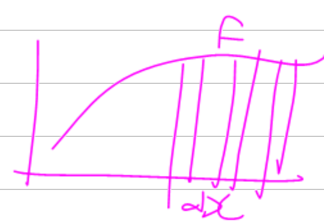
$$\underset{\in \Omega^1(\mathbb{R}^n)}{dx_i}(\xi) = D_\xi x_i = (x_i)' \cdot v$$
$$= v_i = \varphi_i(v)$$

$dx_i(\xi) = \varphi_i(v)$

supersedes φ_i.

$df = \_\, dx_1 + \quad \_\, dx_n$

$$\int_0^1 x^n \, dx = \frac{1}{n+1} \qquad \Delta x$$



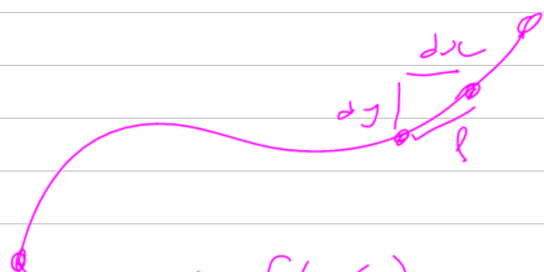
$\Lambda^n \mathbb{R}^n \qquad \Lambda^k \mathbb{R}^n$

$$\mathbb{R}'_x \qquad (dx)(\xi) = v$$

$$\xi = (\underset{\underset{\mathbb{R}}{\cap}}{p}, \underset{\underset{\mathbb{R}}{\cap}}{v})$$

$$y = f(x) \qquad dy = f'(x)\,dx$$

$$\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4} \qquad \frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5} \qquad \frac{\cancel{d}\,\_\_}{d\,\_\,\cancel{x}}$$

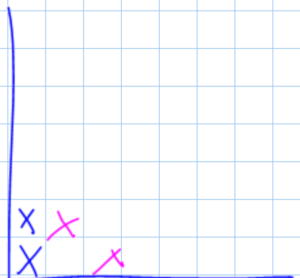


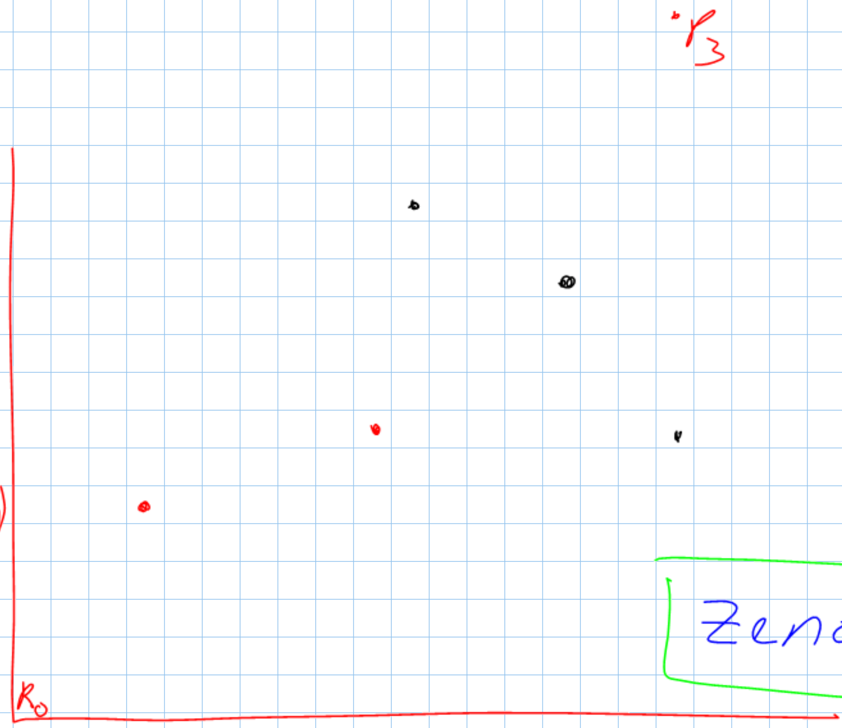
$y = f(x)$

$$dl^2 = dx^2 + dy^2$$

$$l = \int dl = \int \sqrt{dx^2 + dy^2} =$$

$$= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int \sqrt{1 + y'^2}\, dx$$





$R_{k+1} = R_k + (m, n)$

$m, n \geq 0$

$m^2 + n^2 \leq 100$

$\boxed{\text{Zenoish}}$

MAT257 Analysis II on February 12, 2021: d.
Read Along: Spivak 86-92.
Reminder - TA Office Hours: Sebastian Monday 11-12, Shuyang Wednesday 3-4; a great resource!
Reminder - next class in 10 days!
Riddle Along. On Z x Z, a visible roach R starts at (0, 0) and once a minute jumps to the northeast, up to a distance of 10. Meanwhile, an exterminator E can poison one grid point per minute, away from R. Can E trap R?

Diff. k-forms on $\mathbb{R}^n$: $p \mapsto \Lambda^k(T_p\mathbb{R}^n)$ $\quad \Omega^k(\mathbb{R}^n) := \{\text{smooth } k\text{-forms}\}$

$d: \Omega^0(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ by $df(\xi) = D_\xi f = f'(p)\cdot v \quad \xi = (p, v)$

dx_i are φ_i ! so $\lambda \in \Omega^k(\mathbb{R}^n) \Rightarrow \lambda = \sum_{I \in \Delta_n^k} \lambda_I(p)\, dx_I$

Can $+, \alpha\cdot, \wedge$, pull, all nicely compatible. $\quad d(g^*F) = g^*(dF)$

$(F: \mathbb{R}^n \to \mathbb{R}) \in \Omega^0$ $\qquad \mathbb{R}^m \xrightarrow{g} \mathbb{R}^n \xrightarrow{F} \mathbb{R}$

$\xi = (p, v)$

$$(dF)(\xi) = F'(p)\cdot v = \left(\frac{\partial F}{\partial x_1}(p) \; \cdots \; \frac{\partial F}{\partial x_n}(p)\right)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{i=1}^n \frac{\partial F}{\partial x_i}\cdot v_i$$

Conclusion

$$dF = \sum_{i=1}^n \frac{\partial F}{\partial x_i}\, dx_i \qquad = \sum_{i=1}^n \frac{\partial F}{\partial x_i}\, dx_i(\xi)$$

Example

$$\mathbb{R}^2_{r,\theta} \xrightarrow{\phi} \mathbb{R}^2_{x,y} \qquad dx \wedge dy$$

Q $\phi^*(dx \wedge dy)$

$$\phi\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \qquad x = r\cos\theta \quad y = r\sin\theta$$

$$(\phi^*(dx \wedge dy))(\xi_1, \xi_2) =$$

$$= (dx \wedge dy)(\phi_*\xi_1, \phi_*\xi_2) = \cdots\cdots = \text{Ans.}$$

$$x \circ \phi = r\cos\theta$$
$$y \circ \phi = r\sin\theta$$

$$\phi^*(dx \wedge dy) = \phi^*(dx) \wedge \phi^*(dy) = d(\phi^* x) \wedge d(\phi^* y)$$

$$= d(r\cos\theta) \wedge d(r\sin\theta) \qquad \cancel{F}OI\cancel{L}$$

$$= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)$$

$$= \underbrace{r\cos^2\theta\, dr \wedge d\theta}_{O} - \underbrace{r\sin^2\theta\, d\theta \wedge dr}_{I}$$

$$= (r\cos^2 + r\sin^2\theta)\, dr \wedge d\theta = r\, dr \wedge d\theta$$

<u>Def</u> $d : \underset{\omega}{\underset{\cup}{\Omega^k}}(\mathbb{R}^n) \longrightarrow \Omega^{k+1}(\mathbb{R}^n)$ by

$$d\omega = \text{“}\sum_{i=1}^{n} dx_i \wedge \frac{\partial \omega}{\partial x_i}\text{”}$$

Namely, $\omega = \sum_{I \in \Omega_n^k} a_I\, dx_I$

$$d\omega = \sum_{i=1}^{n} \sum_{I \in \Omega_n^k} dx_i \wedge \frac{\partial a_I}{\partial x_i} dx_I$$

<u>Example</u> $\omega = x\, dy \in \Omega^1(\mathbb{R}^2_{x,y})$

$$d\omega = dx \wedge \frac{\partial \omega}{\partial x} + dy \wedge \frac{\partial \omega}{\partial y}$$

$$= dx \wedge dy + dy \wedge 0 \cdot dy = dx \wedge dy$$

$$d(x\,dx) = dx \wedge \underbrace{\frac{\partial x}{\partial x}}_{1} dx + dy \wedge \underbrace{\frac{\partial x}{\partial y}}_{0} dx$$

$$= 0.$$

<u>Properties</u> 1. Linear $d(w_1 + w_2) = \ldots$

$d(\alpha w) = \ldots$ $\square$

2. $d(w \wedge \eta) = dw \wedge \eta + (-1)^{\deg w} w \wedge d\eta$

Leibnitz law

3. $d(g^* w) = g^*(dw)$

4.
$$\begin{tikzcd} \Omega^k \arrow[r, "d"'] \arrow[rr, bend right, "0"'] & \Omega^{k+1} \arrow[r, "d"'] & \Omega^{k+2} \end{tikzcd}$$

$d \circ d = 0$

$d^2 = 0$

0. At $k=0$ $d_{new} f = d_{old} f$. $\square$

<u>Pf of 2</u>

$$\frac{\partial w \wedge \eta}{\partial x_i} = \frac{\partial w}{\partial x_i} \wedge \eta + w \wedge \frac{\partial \eta}{\partial x_i}$$

indeed if $w = f \cdot dx_I$ & $\eta = g \cdot dx_J$

$$w \wedge \eta = f \cdot g \cdot dx_K \qquad K = \begin{cases} \pm\, I \cup J & I \cap J = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial(w\wedge\eta)}{\partial x_j} = \frac{\partial(f\cdot g)}{\partial x_i}\cdot dx_K = \left(\frac{\partial f}{\partial x_j}\cdot g + f\cdot\frac{\partial g}{\partial x_j}\right)dx_K$$

$$d(w\wedge\eta) = \sum dx_i \wedge \frac{\partial(w\wedge\eta)}{\partial x_i}$$

$$= \sum dx_i \wedge \left(\frac{\partial w}{\partial x_i}\wedge\eta + w\wedge\frac{\partial\eta}{\partial x_i}\right)$$

$$= \sum \underbrace{dx_i \wedge \frac{\partial w}{\partial x_i}}_{dw}\wedge\eta + (-1)^{\deg w}\sum w\wedge \underbrace{dx_i\wedge\frac{\partial\eta}{\partial x_i}}_{d\eta}$$

$$= dw\wedge\eta + (-1)^{\deg w} w\wedge d\eta$$

Delts props 2 & 3

What does it mean?