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MAT257 Analysis II on February 1, 2021: More on alternating Tensors: wedge products, minors, pullbacks, degrees 0 and 1, degree n and orientations, degree n-1 and cross products.
Mon -> Mon HW schedule: maybe later. Zoom recording on Friday failed :(.
Read Along: Spivak 75-85.
Riddle Along: Can you colour the points of the plane with 3 colours, so that no two points of distance exactly 1 have the same colour? How about 4, 5, 6, or 7?
Riddled before: Is there a continuous surjection f:[0,1]->[0,1] which is constant on a set of

$$\int_C dw = \int_{\partial C} w$$

$\langle v_i \rangle = V$

$\langle \varphi_j \rangle = V^*$

$$\Lambda^k(V) = \langle w_I \rangle_{I \in \Omega_n^k} \quad \left(\dim = \binom{n}{k}\right)$$

$$\Omega_n^k = \{1 \le i_1 < \dots < i_k \le n\} \qquad w_I = \sum_{\sigma \in S_k} (-1)^\sigma \varphi_I \circ \sigma^*$$

Thm $\exists!$ family of bilinear ops

$$\wedge : \Lambda^k(V) \times \Lambda^l(V) \longrightarrow \Lambda^{k+l}(V)$$

$$w \quad \lambda \longmapsto w \wedge \lambda$$

s.t. 1. Associative $\underset{\Lambda^k}{(w} \wedge \underset{\Lambda^l}{\lambda)} \wedge \underset{\Lambda^m}{\eta} = w \wedge (\lambda \wedge \eta)$ in Λ^{k+l+m}

2. $\wedge$ is "graded-commutative"
"super-commutative"

$$w \wedge \lambda = (-1)^{kl} \lambda \wedge w \qquad w \in \Lambda^k, \ \lambda \in \Lambda^l$$

$$= (-1)^{(\deg w)\cdot(\deg \lambda)} \lambda \wedge w$$

$$= (-1)^{|w||\lambda|} \lambda \wedge w$$

3. $w_I = \varphi_{i_1} \wedge \varphi_{i_2} \wedge \varphi \dots \wedge \varphi_{i_k}$ } depends on choosing a basis

Pf Uniqueness $\varphi_i \wedge \varphi_j \overset{(2)}{=} -\varphi_j \wedge \varphi_i$

$$\varphi_i \wedge \varphi_i = -\varphi_i \wedge \varphi_i = 0$$

$$\underset{|I|=k}{\omega_I} \wedge \underset{|J|=l}{\omega_J} \overset{(3)}{=} (\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}) \wedge (\varphi_{j_1} \wedge \dots \wedge \varphi_{j_l})$$

$$\overset{(1)}{=} \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k} \wedge \varphi_{j_1} \wedge \dots \wedge \varphi_{j_l}$$

$$= \begin{cases} \pm \omega_M & \text{if } \forall \alpha, \beta \ \ i_\alpha \neq j_\beta \\ 0 & \text{if } \exists \alpha, \beta \ \ i_\alpha = j_\beta \end{cases}$$

where $m_1 \dots m_{k+l}$ are $i_1 \dots i_k, j_1 \dots j_l$ sorted to be ascending

$$\omega_{125} \wedge \omega_{347} = (-1)^2 \omega_{123457}$$

$$125345 \xrightarrow[\text{2 flips}]{} 12345$$

$$\lambda \wedge \eta = \underbrace{\left(\sum l_I \omega_I\right)}_{\lambda} \wedge \underbrace{\left(\sum h_J \omega_J\right)}_{\eta}$$

$$= \sum_{I,J} l_I h_J \cdot \omega_I \wedge \omega_J$$

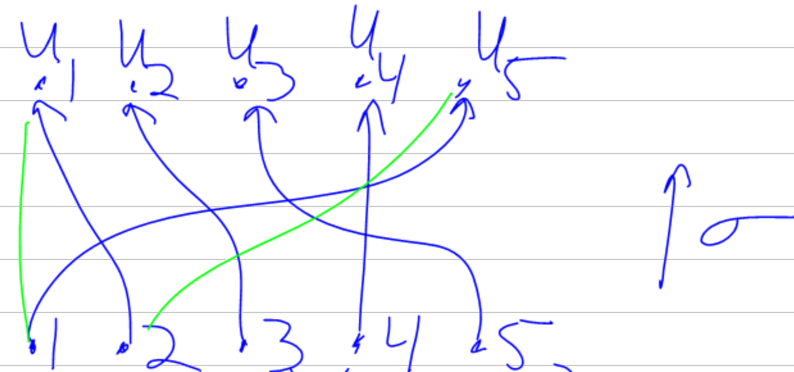
Existence: Given $\lambda \in \Lambda^k, \eta \in \Lambda^l$

$$(\lambda \wedge \eta)(u_1 \dots u_{k+l})$$

$$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma \lambda(u_{\sigma 1}, \ldots u_{\sigma k})\, \eta(u_{\sigma(k+1)}, \ldots, u_{\sigma(k+l)})$$

$\lambda \in \Lambda^3$

$\eta \in \Lambda^2$



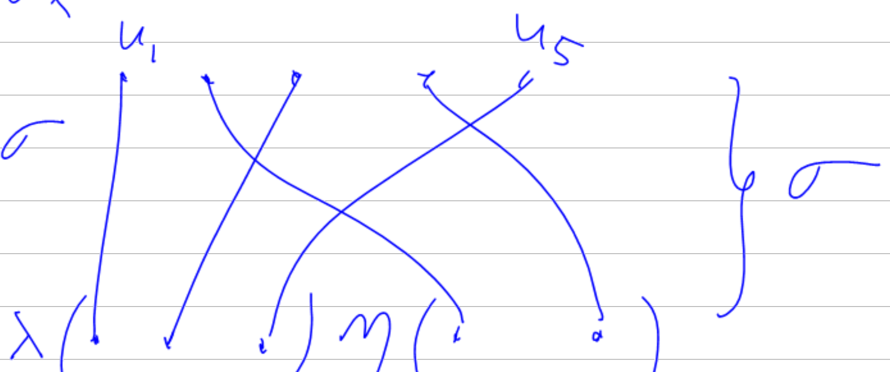
$$\frac{1}{3!\,2!} (-1)^\sigma \quad \lambda(u_{\sigma 1}\ u_{\sigma 2}\ u_{\sigma 3})\, \eta(u_{\sigma 4}\ u_{\sigma 5})$$

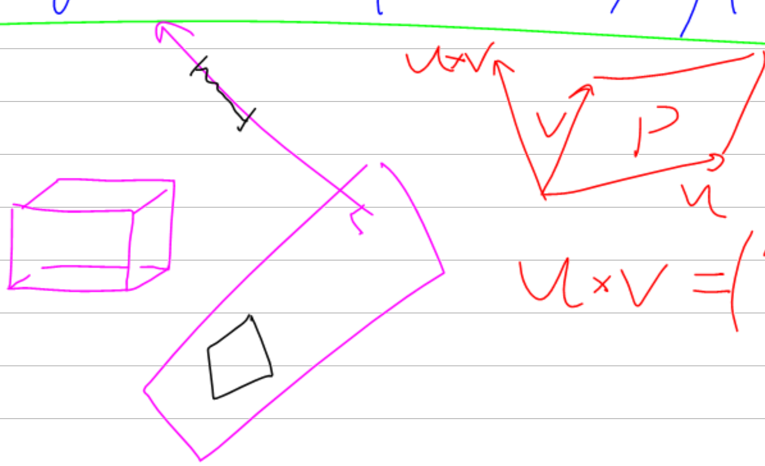
$u_{\sigma 1} < u_{\sigma 2} < u_{\sigma 3}$, $u_{\sigma 4} < u_{\sigma 5}$

* This really is alternating $\in \Lambda^{k+l}$

$$= \sum_{\substack{\sigma \in S_{k+l} \\ \sigma 1 < \sigma 2 < \ldots < \sigma k \\ \sigma(k+1) < \sigma(k+2) < \ldots < \sigma l}} (-1)^\sigma \lambda(u_{\sigma 1} \ldots u_{\sigma k})\, \eta(u_{\sigma(k+1)} \ldots u_{\sigma l})$$

$$\lambda \wedge \eta : \sum_{\text{such } \sigma} (-1)^\sigma$$





$$u \times v = \begin{pmatrix}\text{area} \\ \text{of} \\ P\end{pmatrix} \begin{pmatrix}\text{a vect} \\ \perp \text{ to} \\ \text{both}\end{pmatrix}$$

MAT257 Analysis II on February 3, 2021: Wedge products, pullbacks, volume elements and orientations.
HW13 (a tough one!) is on the web. Mon -> Mon HW schedule: maybe later.
Read Along: Spivak 75-85.
Riddled before: Is there a continuous surjection f:[0,1]->[0,1] which is constant on a set of intervals whose lengths sum to 1?
Riddled before: A unit cube in R^3, the area of its projection on any plane is equal to the length of its projection on a perpendicular line to that plane.

<u>Thm</u> $\underset{\text{nts}}{\exists}\,\underset{\text{done}}{!}$ $\left(\underset{\Lambda^k}{\lambda}, \underset{\Lambda^l}{\eta}\right) \longmapsto \underset{\Lambda^{k+l}}{\lambda\wedge\eta}$ s.t.

1. Assoc. $(\lambda\wedge\eta)\wedge\phi = \lambda\wedge(\eta\wedge\phi)$

2. Super-commutative: $(\lambda\wedge\eta) = (-1)^{kl}\eta\wedge\lambda$

3. $\omega_I = \varphi_{i_1}\wedge\dots\wedge\varphi_{i_k}$

<u>Existence</u> Set $(\lambda\wedge\eta)(u_1,\dots,u_{k+l}) = \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} (-1)^\sigma (\lambda\otimes\eta)(\sigma^*(u_1,\dots,u_{k+l}))$

$\lambda\wedge\eta\in\Lambda^{k+l}(V)$

$$= \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} (-1)^\sigma \lambda\left(\boxed{u_{\sigma 1},\dots,u_{\sigma k}}\right)\eta\left(\boxed{u_{\sigma(k+1)},\dots,u_{\sigma(k+l)}}\right)$$

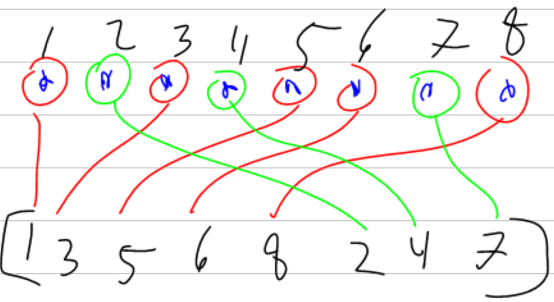
$$= \sum_{\substack{\sigma\in S_{k+l}\\ \sigma 1<\dots<\sigma k\\ \sigma(k+1)<\dots<\sigma(k+l)}} (-1)^\sigma \lambda(u_{\sigma 1},\dots,u_{\sigma k})\,\eta(u_{\sigma(k+1)},\dots,u_{\sigma(k+l)})$$

$k=3$ $l=2$:

$$\sum_{\text{"good"}} (-1)^\sigma \lambda(\_\ \_\ \_)\,\eta(\_\ \_)$$

"$\wedge$ splitting"

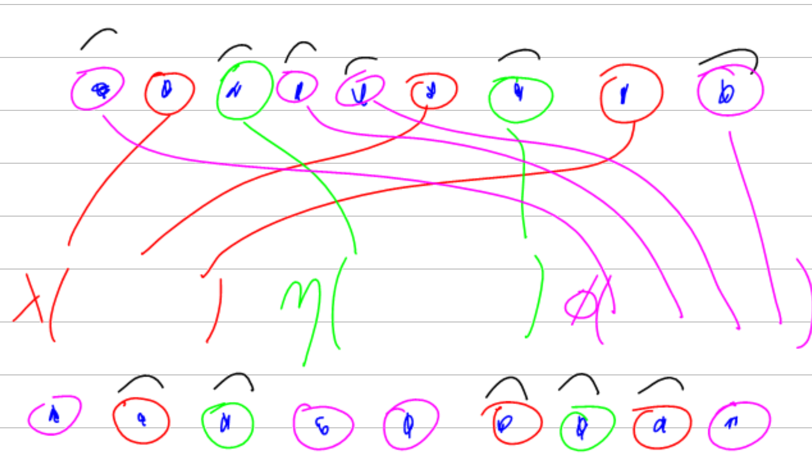
$\sigma = [2\,4\,5\,1\,3]$
1 2 3



[1 3 5 6 8 2 4 7]

<u>Pf of Assoc.</u> $k=3$ $l=2$ $m=4$ $k+l+m=9$

$\underset{k}{\lambda}\wedge\underset{l}{\eta}\wedge\underset{m}{\phi}$



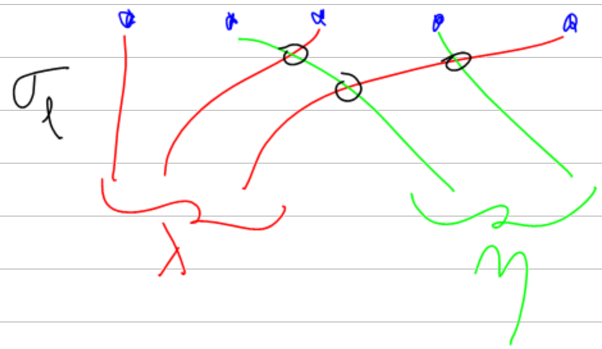
$\underset{3}{\lambda}\wedge\underset{6}{(\eta\wedge\phi)}$

$\underset{5}{(\lambda\wedge\eta)}\wedge\underset{4}{\phi}$

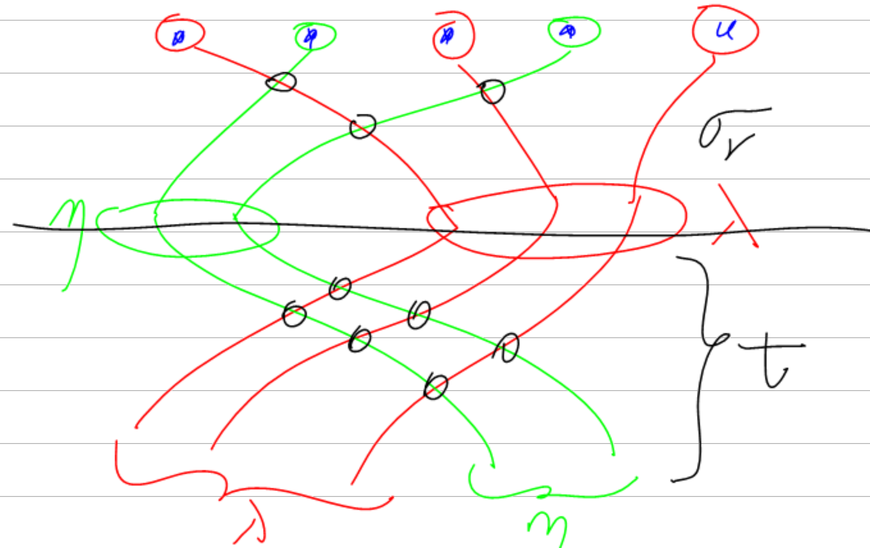
$\lambda \wedge \eta = (-1)^{kl} \eta \wedge \lambda$

3 2

lhs:



rhs



$\text{sign}\, \tau = (-1)^{k \cdot l}$

3. $\psi_I = \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$ [left to you]

Pullbacks. $V \xrightarrow{L} W$ L: Linear trans.

$$\Lambda^k(V) \xleftarrow{L^*} \Lambda^k(W)$$

$$\underset{\substack{\vee\\ \psi}}{\Lambda^k(W)} \ni \lambda$$

$$(L^*\lambda)(u_1, \dots, u_k) := \lambda(Lu_1, \dots, Lu_k), \quad u_1, \dots, u_k \in V$$

__Prop__ 1. $L^*(a\lambda + b\eta) = \dots$

2. $L^*(\lambda \wedge \eta) = (L^*\lambda) \wedge (L^*\eta)$

$V \quad \dim V = n$

$$\underbrace{\Lambda^n(V)}_{\substack{1\text{-dim} \\ \binom{n}{n}=1}} = \Lambda^{top}(V) = \left\{ \begin{matrix} \text{Volume} \\ \text{forms} \\ \text{on } V \end{matrix} \right\}$$

$$\begin{matrix} \Lambda^{top}(V) \\ \cup \\ \lambda \end{matrix}$$

$$\lambda(u_1, \ldots u_n) \cancel{\notin} \in \mathbb{R}$$

$$\eta \in \Lambda^k(V) \qquad \eta(u_1 \ldots u_k) \in \mathbb{R}$$

$$V \xrightarrow{L} V \qquad M(L) = A \in M_{n \times n}(\mathbb{R})$$

$$\Lambda^{top}(V) \underset{\text{scalar?}}{\overset{L^*}{\longleftarrow}} \begin{matrix} \Lambda^{top}(V) \\ \cup \\ w \end{matrix}$$

<u>claim</u> $L^* w = (\det(A)) w \qquad L^* w = \delta(A) \cdot w$

<u>Def of det</u> $\det : M_{n \times n} \longrightarrow \mathbb{R}$ unique s.t.

1. multilinear in cols.
2. Known behavior over col ops.
3. $\det(I) = 1$.

MAT257 Analysis II on February 5, 2021: Orientations, tangent vectors.
Read Along: Spivak 82-92.
Riddled before: Is there a continuous surjection f:[0,1]->[0,1] which is constant on a set of intervals whose lengths sum to 1?
Riddled before: A unit cube in R^3, the area of its projection on any plane is equal to the length of its projection on a perpendicular line to that plane.

V n-dim $\quad \Lambda^n(V) = \Lambda^{top}(V) = \{$volume forms on $V\}$

A non-zero volume form, considered up to mult by a positive scalar, $\omega \sim 7\omega \sim \frac{7}{22}\omega \not\sim -\omega$ is called "an orientation" of V.

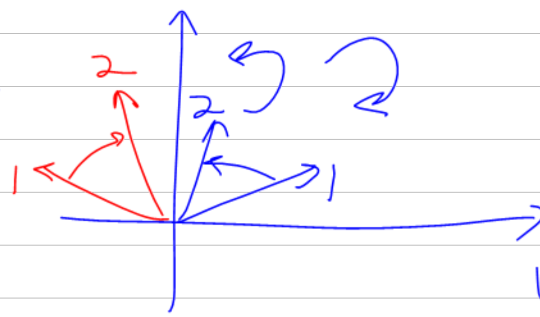
Every V has two orientations

$\Lambda^{top} \ni \omega \neq 0$ & $-\omega$

Why? Examples 1. 1D

basis up to something.

2.



orientation is a choice of an ordered basis, up to something

3. "right hand rules"
"left hand rules"
A choice of a basis, up to something.

Def An orientation of an n-dim

V.S. is a choice of an ordered basis $(v_1 \ldots v_n)$ but we say that $(v_1 \ldots v_n) \sim (u_1 \ldots u_n)$
↑ "same orientations"

if the change of basis matrix between them has positive det.

<u>Claim</u> "orientation" $\Longleftrightarrow$ "orientation"

Vol form up to a pos. scalar $\Leftarrow$ basis up to pos. det C.O.B.

$$\psi$$

$$W \qquad\qquad v = (v_1 \ldots v_n)$$

We will say that "w agrees with v"
if $w(v) > 0$ otherwise they are "opposite"

Likewise $u_i = \sum a_{ij} v_j \quad A = (a_{ij})$

$$\det A > 0$$

$$w(u) = \underbrace{(\det A)}_{>0} w(v) \qquad \text{(Exercise)}$$

$\Rightarrow$ v agrees w/ w iff u agrees w/ w.

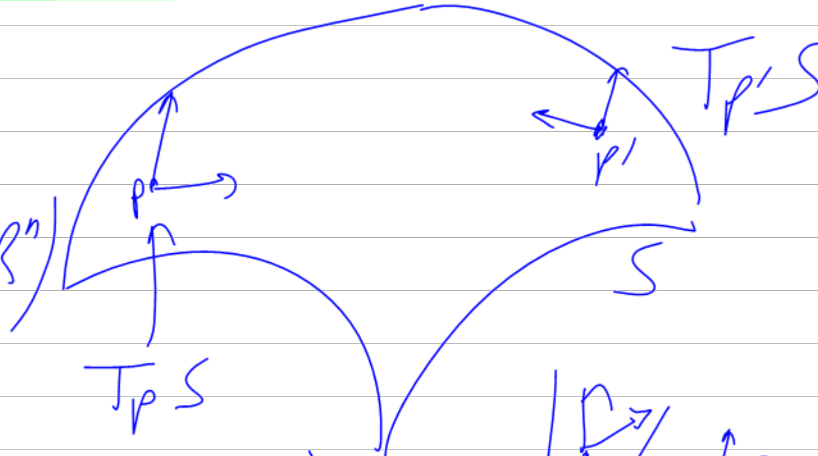
$$V \xrightarrow[\text{invertible}]{L} W$$

both push & pull but only if L

$V=(V_1 \dots V_n) \longmapsto L_* V = (LV_1 \dots LV_n)$ is invertible

$L^* W \longleftarrow W$

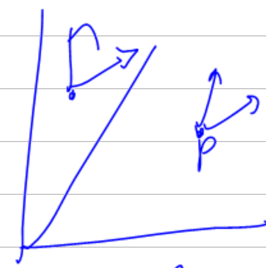
Tangent vectors/spaces



<u>Def</u> Given $\vec{V}$ $(\sim \mathbb{R}^n)$

& $p \in \vec{V}$

$$T_p V = \vec{V}_p = \left(T_p\mathbb{R}^n = \mathbb{R}^n_p\right)$$



$$= \{(p, v) \sim v_p : v \in \vec{V}\} \quad \left(\text{clearly equiv } \vec{V}\right)$$

is a vector space

$$(p, v_1) + (p, v_2) \neq (2p, v_1 + v_2) = (p, v_1 + v_2)$$

$$\alpha(p, v) = (p, \alpha v)$$

$$0_{T_p(V)} = (p, 0)$$

If $\vec{V}$ has an inner product, so does $T_p\vec{V}$

$$\langle (p, v_1), (p, v_2) \rangle := \langle v_1, v_2 \rangle$$

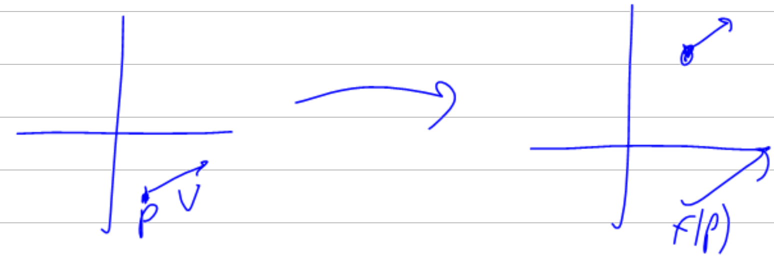
Push or pull?

Suppose $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is diff'able.

$F_* : T_p\mathbb{R}^n \longrightarrow T_{F(p)}\mathbb{R}^m$

by

$F_*(p, v) = (F(p), F'(p) \cdot v)$



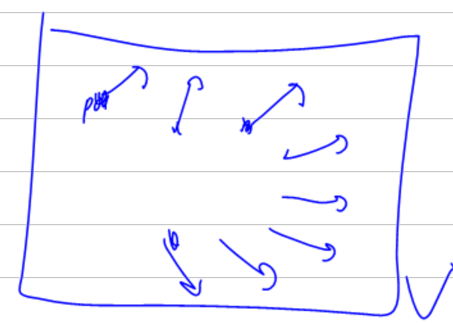
<u>Def</u> a function $F : V \longrightarrow \bigcup_p T_p V$

$p \in V$

s.t.

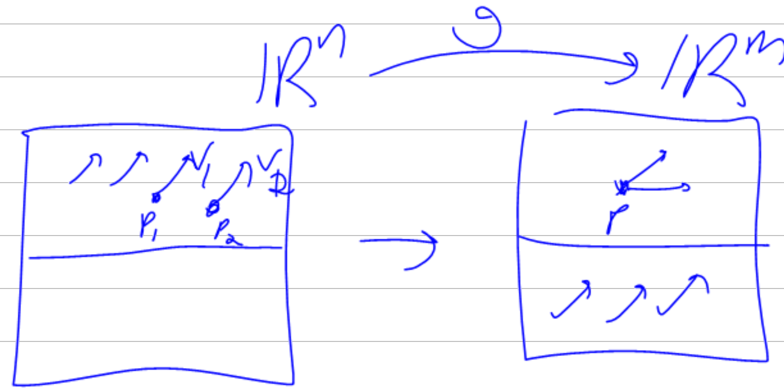
$F(p) \in T_p V$

is called "a vector field"



$F, G \longmapsto F + G$

αF



Push or pull?

Ans: Neither