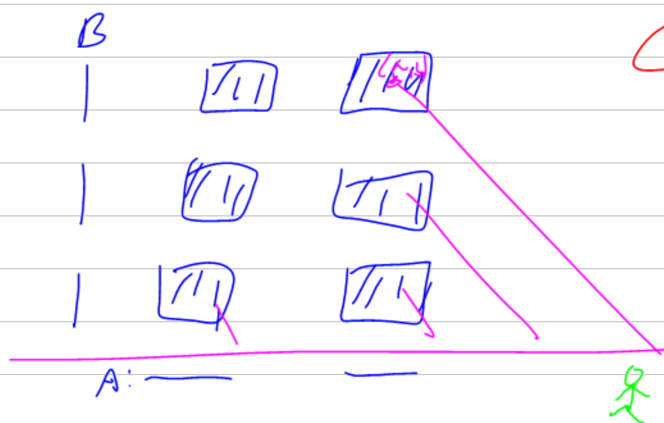
$$
\begin{array}{r}
0.002 \\
0.002 \\
\hline
0.0121011
\end{array}
\qquad
\begin{array}{c}
22 \\
02 \\
\cancel{2}1
\end{array}
$$

$$C+C=[0,2]$$

$$A+B=\{a+b : a\in A,\ b\in B\}$$

$C\times C\subset\mathbb{R}^2$ is ~~dust~~ just

Yet, does it have a shadow?

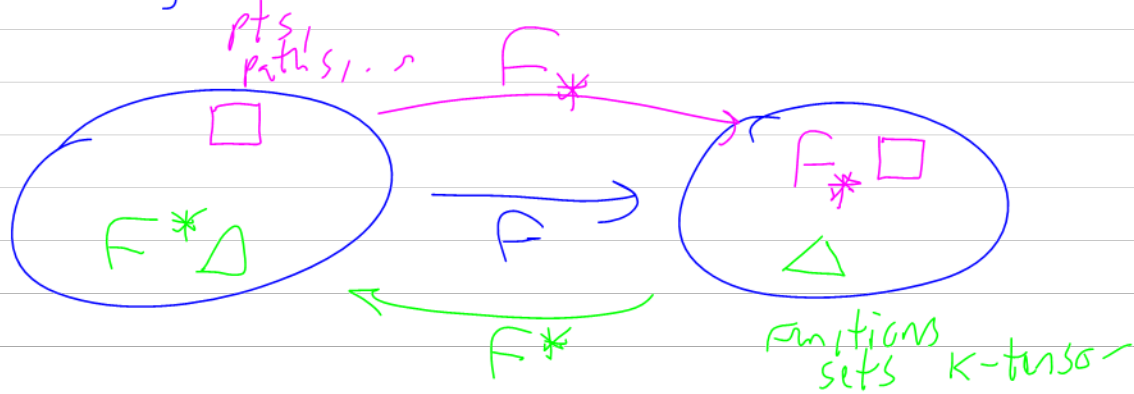


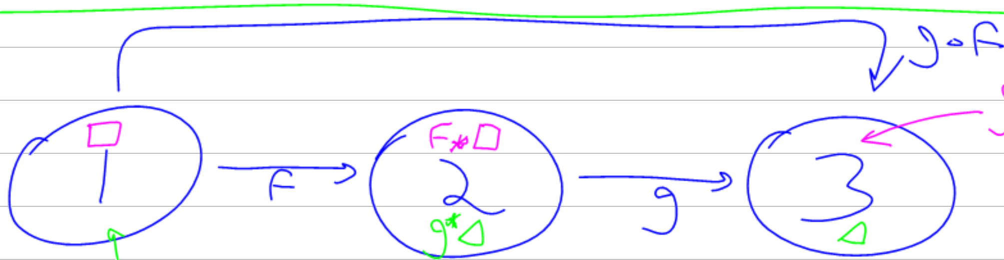
MAT257 Analysis II on January 25, 2021: Alternating Tensors.
TT2: Not yet.
Read Along: Spivak 75-85.
Riddle Along: The Cantor set C is of measure 0. Is the same true for C+C={x+y: x,y in C}?

$\underline{n} = \{1, \dots, n\}$ $\quad v_i, \varphi_i \quad \varphi_i(v_j) = \delta_{ij}$

Recall: $I, J \in \underline{n}^k$, $\varphi_I = \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ $\quad v_I = (v_{i_1}, \dots, v_{i_k})$ $\quad \varphi_I(v_J) = \delta_{IJ}$

$\Rightarrow \{\varphi_I\}$ is a basis of $\mathcal{T}^k(V)$; $\dim \mathcal{T}^k(V) = n^k$

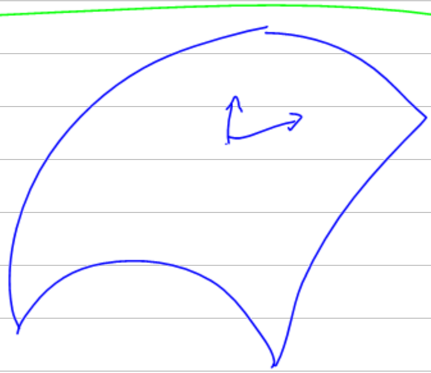




$g_*(f_*\square) \overset{?}{=} (g \circ f)_*\square$!!

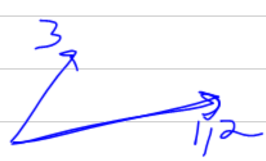
$f^*(g^*\triangle) \overset{?}{=} (g \circ f)^*\triangle$!!

Exercise: Verify covariance/contravariance For each of pts, paths, sets, ... & more.



Suppose $T(u_1 \dots u_k)$ is to measure "volume". Then we expect

$$T(\dots u \dots u \dots) = 0 \quad (*)$$



Def $\Lambda^k(V) \subset \mathcal{T}^k(V)$

Those k-tensor satisfying (*).

Claim $T \in \mathcal{T}^k(V)$ "kills repetitions" $(T(\ldots u \ldots u \ldots) = 0)$ iff T is "alternating" $\left(T(\ldots u \ldots v \ldots) = -T(\ldots v \ldots u \ldots) \right)$

Pf $\Leftarrow$: Suppose T is alternating

$T(\ldots u \ldots, u \ldots) = -T(\ldots u \ldots u \ldots)$

$\Rightarrow T(\ldots u \ldots u \ldots) = 0$.

$\Rightarrow$: Suppose T kill repetitions,

$T(\ldots u+v \ldots, u+v \ldots) \rightsquigarrow T(u+v, u+v)$

$0 = T(u+v, u+v) = T(u, u+v) + T(v, u+v)$

$= \underbrace{\cancel{T(u,u)}}_{0} + T(u,v) + T(v,u) + \underbrace{\cancel{T(v,v)}}_{0}$

$\Rightarrow T(u,v) + T(v,u) = 0 \Rightarrow T$ is alter.

Def $\Lambda^k(V) := \{ w \in \mathcal{T}^k(V) : w \text{ is alternating} \}$

"Alternating k-tensors"

If $w \in \Lambda^k(V)$ $w(\ldots u, \ldots, 7u \ldots) =$
$7 w(\ldots u, \ldots, u \ldots) = 7 \cdot 0 = 0$.

Aside on permutations

$$S_K = \{\text{all permutations of } \{1, \ldots, K\}\}$$

$$= \{\text{bijections } \pi : \underline{K} \longrightarrow \underline{K}\}$$

$$\Updownarrow$$

$$|S_K| = K! \qquad [\pi 1, \pi 2, \pi 3, \ldots, \pi K]$$

$$K \cdot (K-1) \cdot (K-2) \cdots 1 = K!$$

IF $\pi, \sigma \in S_K$ $\quad \pi \circ \sigma : \underline{K} \longrightarrow \underline{K}$

$$\Rightarrow \pi \circ \sigma \in S_K$$

$$\pi^{-1} \in S_K \qquad \pi \circ \pi^{-1} = \iota \in S_K$$

$$\iota 1 = 1 \quad \iota 2 = 2 \ldots$$

1. $(\pi \circ \sigma) \circ \lambda = \pi \circ (\sigma \circ \lambda)$
2. $\iota \circ \pi = \pi = \pi \circ \iota$
3. $\pi \circ \pi^{-1} = \iota$

347: S_K is a "group"

$\cancel{4}$ $\quad \pi \circ \sigma \neq \sigma \circ \pi$

"non-commutative"

<u>Thm</u> $\exists !$ $\text{sign} : S_K \longrightarrow \{\pm 1\}$ s.t.

1. $\text{sign}(\pi \circ \sigma) = \text{sign}(\pi) \cdot \text{sign}(\sigma)$
2. IF τ is a transposition: there are $i \neq j \in \underline{K}$ s.t.

$$\tau \ell = \begin{cases} j & \ell = i \\ i & \ell = j \\ \ell & \text{otherwise} \end{cases}$$

1 2 6 4 5 3 7 8 9 $\qquad k = 9$

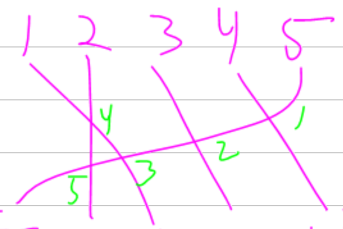
then $\text{sign}(\tau) = -1$.

Alt. notation $\quad \text{sign}(\pi) = (-1)^{\pi} = (-1)^{\pi}$

$\text{sign}(\pi) = 1 \iff$ "π is even"

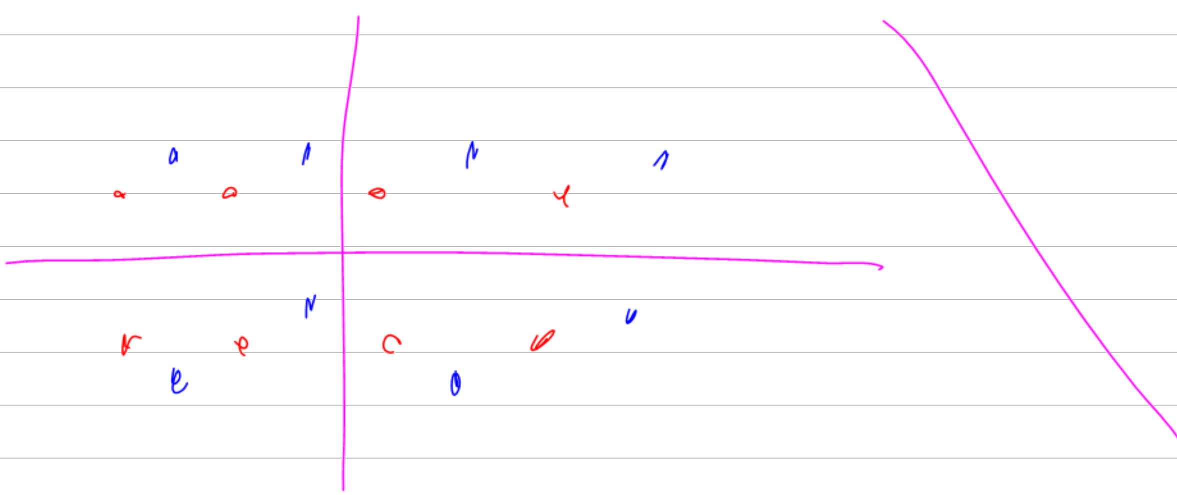
$\text{sign}(\pi) = -1 \iff$ "π is odd".

"PF" (mostly see 240/247/347)

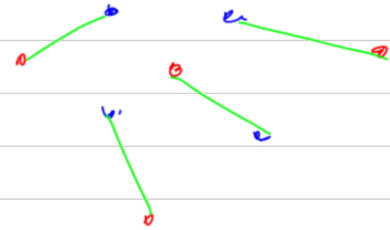


$\pi = [5\ 2\ 1\ 3\ 4]$

$\text{sign}(\pi) = (-1)^5 = (-1)$.



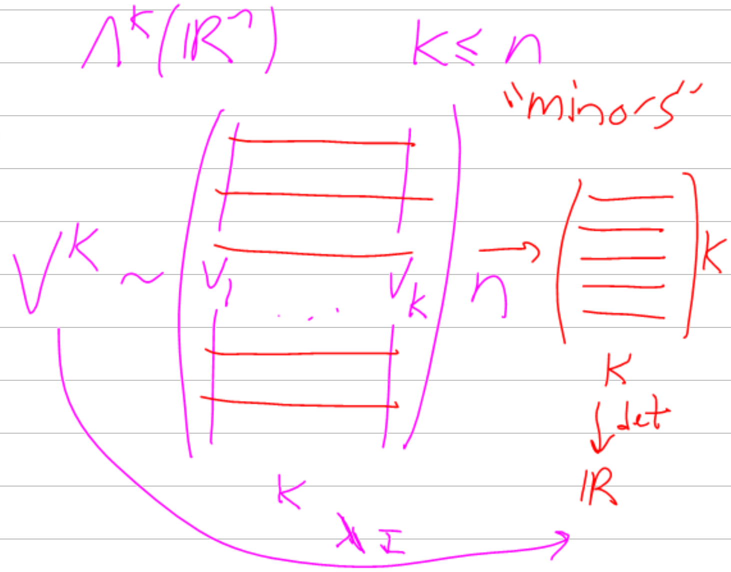
MAT257 Analysis II on January 27, 2021: More on alternating Tensors.
Read Along: Spivak 75-85.
TT2 results sent! Appeals deadline Friday Feb 5 at 5PM.
HW12 is on Crowdmark!
Riddle Along: n red points and n blue points are placed in the plane with no 3 on the same line. Prove that it is possible to pair them up using n straight line segments so that no two of the segments will intersect.



Def $\Lambda^k(V) = \{T \in \mathcal{T}^k(V) : T \text{ is alternating / anti-symmetric}\}$

A subspace!

Examples: Determinants & minors!



$S_k = \{\text{bijections } \underline{k} \to \underline{k}\}$

"The k-th symmetric/permutation group"

Thm $\exists!$ $\text{sign}: S_k \to \{\pm 1\}$ s.t. $\sigma \mapsto (-1)^\sigma$

$(-1)^{\sigma\circ\lambda} = (-1)^\sigma(-1)^\lambda, \quad (-1)^{\tau_{ij}} = -1$

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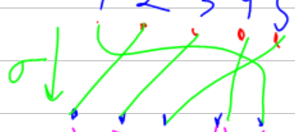
3 Formulas:

① 1 2 3 4 5



1 2 3 4 5 $\in S_5$

2 3 5 4 1

$\text{sign} = (-1)^5 = (-1)$

② $\left(\sum_{1 \le i < j \le n} \begin{cases} 0 & \sigma i < \sigma j \\ 1 & \sigma i > \sigma j \end{cases} \right)$

$\text{sign}(\sigma) = (-1)$

$= \prod_{1 \le i < j \le n} \text{sign}(\sigma j - \sigma i)$

③ $\sigma \longmapsto P_\sigma = \begin{pmatrix} & & 1 \\ 1 & & 0 \\ & & 0 \\ 1 & & 0 \\ & & 0 \end{pmatrix} \leftarrow \sigma n$

$(P_\sigma)_{ij} = \delta_{i,\sigma j}$

$\text{sign}(\sigma) = \det(P_\sigma)$ $\quad P_{\sigma\circ\lambda} = P_\sigma \cdot P_\lambda$

$\lambda \in \Lambda^k(V)$ basis? dim?

Choose a basis $v_1 \dots v_n$ of V

Let (φ_j) be the dual basis of V^*

$\varphi_i(v_j) = \delta_{ij}$

$\lambda(u_1, \dots, u_k) = \dots = \lambda(v_{i_1}, \dots v_{i_k})$ comb of

$\lambda(v_3, v_1, v_2) = (-1)^2 \lambda(v_1, v_2, v_3)$

comb of

$\lambda(v_{i_1}, v_{i_2}, \dots v_{i_k})$

$\underbrace{1 \le i_1 < i_2 < i_3 \dots < i_k \le n}_{I}$

I is "ascending" $\quad$ 1 2 3 4 5 6 7 $\quad$ $n=7$, $k=3$

$T \in \mathcal{T}^k$ is determined by $T(\underbrace{v_{i_1}, v_{i_2} \dots v_{i_k}}_{v_I})$

$I = (i_1 \dots i_k)$

$T(v_I)$

$$I \in \mathcal{I}^k_a = \{(i_1 \dots i_k) : 1 \le i_1 < i_2 \dots < i_k \le n\}$$

Conjecture That's all, $\dim \Lambda^k(\underset{n\text{-dim}}{V}) = |\mathcal{I}^k_a| = \binom{n}{k}$

need: $w_I \in \Lambda^k(V)$ $\quad I \in \mathcal{I}^k_a$

$w_I(v_J) = \delta_{IJ}$ $\quad I, J$

guess $w_I = \varphi_I$

make this alt?

v_I $\quad \varphi_I \in \mathcal{T}^k(V)$

$\varphi_I(v_J) = \delta_{IJ}$

span $\quad T = \sum c_I \varphi_I \quad c_I = T(v_I)$

LinInd $\quad \sum c_I \varphi_I = 0$ / ev on v_J

$\Downarrow$

$c_J = 0$

$f(x,y) = -f(y,x)$

g doesn't!

set $f(x,y) = g(x,y) - g(y,x)$

$$w_I(u_1 \ldots u_k) = \sum_{\sigma \in S_k} (-1)^\sigma \varphi_I(u_{\sigma 1}, u_{\sigma 2}, \ldots u_{\sigma k})$$

<u>Claim</u> 1. $w_I \in \Lambda^k(V)$

2. $w_I(v_J) = \cancel{\delta_{IJ}}\, \delta_{IJ} \quad I, J \in \Omega^k_n$

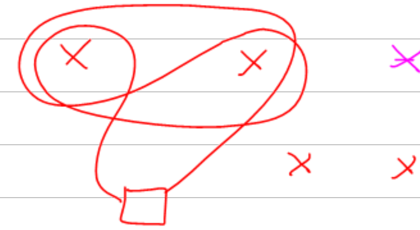
3 hence conj is proven!

$\{w_I\}$ are a basis $\Rightarrow \dim \Lambda^k(V) = \binom{n}{k}$.

$$\begin{tikzcd} L : \underset{(v_i)}{V} \arrow[r] \arrow[dr] & \underset{(w_i)}{W} \arrow[d, "\varphi"] \\ & \mathbb{R} \end{tikzcd} \qquad L^* : \overset{\psi_j}{W^*} \longrightarrow \overset{\varphi_j}{V^*}$$

$$M(L^*)_{\psi, \varphi} = \left(M(L)_{v, w}\right)^T$$

MAT257 Analysis II on January 29, 2021: More on alternating tensors: a basis, an aside, a product.
Read Along: Spivak 75-85.
Zoom recording today!
Monday -> Monday HW cycle?
Riddle Along: Can you hang a medal on 2 nails so that if you remove any one of them, the medal will fall? Same with 3 nails? And 4?



Thm $\exists!$ sign: $S_k \to \{\pm 1\}$, $\sigma \mapsto (-1)^\sigma$ s.t. $(-1)^{\sigma\circ\lambda} = (-1)^\sigma(-1)^\lambda$, $(-1)^{\tau_{ij}} = -1$.

Def $\Lambda^k(V) = \{T \in \mathcal{T}^k(V) : \forall \sigma \in S_k,\ T\circ\sigma^* = (-1)^\sigma T\}$

$\binom{n}{k}$ $\quad \binom{X}{k} = \left\{\begin{matrix}\text{subset of } X \\ \text{of size } k\end{matrix}\right\}$ $\quad \left|\binom{X}{k}\right| = \binom{|X|}{k}$

$\{1 \le i_1 < i_2 < \dots < i_k \le n\} =: \underline{n}^k_a \sim \binom{\underline{n}}{k} \subset \underline{n}^k$

$\sigma : \underline{k} \to \underline{k}$

$\sigma^* : V^k \to V^k$, $u = (u_1, \dots, u_k)$

$(\sigma^* u)_i = u_{\sigma_i}$

$$\begin{tikzcd} \underline{k} \arrow[r, "\sigma"] \arrow[rr, bend right, "\sigma^* u"'] & \underline{k} \arrow[r, "u"] & V \end{tikzcd}$$

$(\sigma\circ\tau)^* = \tau^*\circ\sigma^*$

$\lambda \in \Lambda^k(V)$ is determined by $\lambda(V_I)$, $I \in \underline{n}^k_a$!

$W_I := \sum_{\sigma\in S_k} (-1)^\sigma \varphi_I \circ \sigma^* \qquad I \in \underline{n}^k_a$

Claim $W_I \in \Lambda^k(V)$, $W_I(V_J) = \delta_{IJ}$

hence $\{W_I\}_{I\in\underline{n}^k_a}$ is a basis of $\Lambda^k(V)$, hence $\dim \Lambda^k(V) = \binom{n}{k}$.

Pf $W_I \circ \tau^* = \qquad \tau \in S_k$

$\boxed{f(x,y) = -f(y,x)}$

$\left(\sum_\sigma (-1)^\sigma \varphi_I \circ \sigma^*\right) \circ \tau^*$

$= \sum_\sigma (-1)^\sigma \varphi_I \circ \sigma^* \circ \tau^*$

$= \sum_\sigma (-1)^\sigma (-1)^\tau (-1)^\tau \varphi_I \circ (\tau\circ\sigma)^*$

$= \sum_\sigma (-1)^{\overbrace{\tau\circ\sigma}^{\eta}} (-1)^\tau \varphi_I \circ (\overbrace{\tau\circ\sigma}^{\eta})^*$

$$= (-1)^\tau \underbrace{\sum_\eta (-1)^\eta \varphi_I \circ \eta^*} = (-1)^\tau \cdot W_I$$

$$\Rightarrow W_I \in \Lambda^k(V)$$

$I, J \in \underline{n}^k_\wedge$

$$W_I(V_J) = \sum_\sigma (-1)^\sigma (\varphi_I \circ \sigma^*)(V_J)$$

$$= \sum_\sigma (-1)^\sigma \varphi_I(\underbrace{\sigma^* V_J}_{V_{\sigma_* J}}) = \underbrace{\sum_\sigma (-1)^\sigma \delta_{I, \sigma_* J}}_{\text{only cont. from } \sigma = \tau}$$

$$= \delta_{I,J}$$

<u>Span</u> $\lambda \in \Lambda^k$ $\lambda = \sum c_I W_I$ (wnt)

set $c_I = \lambda(V_I)$ <u>Claim</u> this works.

<u>Lin Ind</u> Assume $\sum c_I W_I = 0$ / evaluate V_J

$c_J = 0$ $\square$ $\binom{3}{7} = 0$

$$\dim \Lambda^k(V) = \begin{cases} \binom{n}{k} & 0 \le k \le n \\ 0 & k > n \end{cases}$$

Aside $\quad \Delta^k_{nd} = \{1 \le i_1 \le i_2 \le i_3 \le \dots \le i_k \le n\}$

$n = 5 \quad k = 7$

$1223555 \longleftrightarrow 1\boxed{*2**3*45***}$

$$|\Delta^k_{nd}| = \binom{n+k-1}{k}$$

Chars in green box: $n+k-1$
of those k are $*$'s.

Next: $\exists \quad \wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$

s.t. 1. $\quad$:

2. $\quad$:

3. $\quad$: