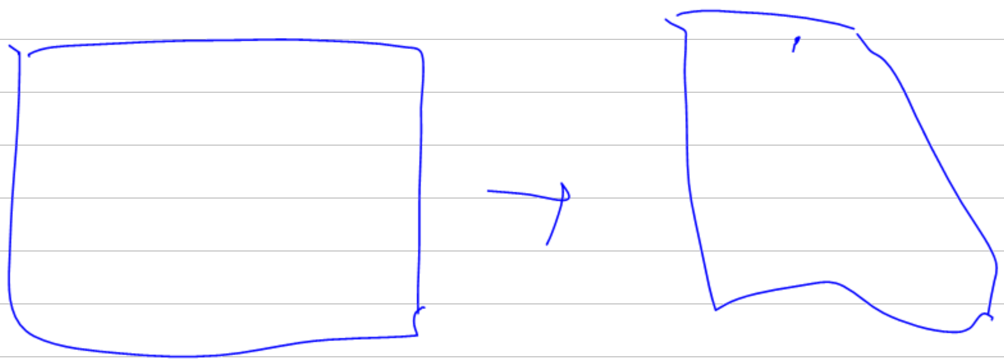


MAT257 Analysis II on January 18, 2021.
Read Along: Spivak 75-85.
TT2 tomorrow, 5-7PM.
Office hours today: Sebastian 11-2. Tomorrow Shuyang 10:30-11:30, and Dror 9-10 and 12-1.
Please be gentle with zoom chat!
* Class-related only.
* Try not to reach the point where the chat is a distraction.
* Be gentle! Be kind! You don't see everyone - your words may offend somebody you don't even see.

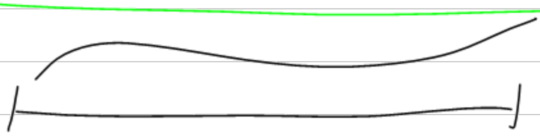
Riddle Along: Can you fold a rectangular piece of paper (perhaps many times) so that the result will have a longer perimeter than the original?

Remember,

$$\int_C d\omega = \int_{\partial C} \omega$$

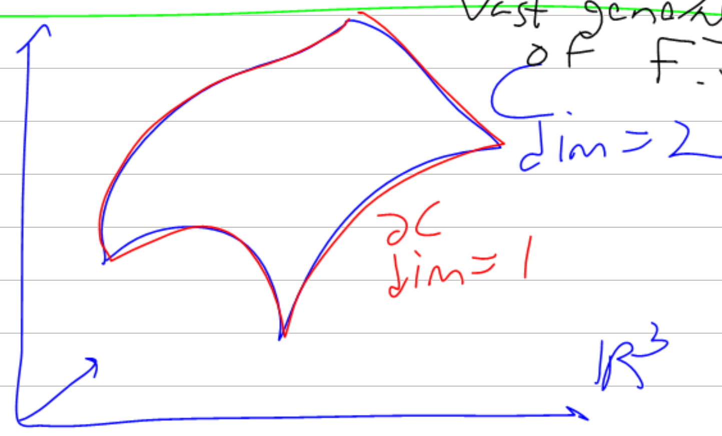
Stokes' thm

Vast generalization of F.T.C.

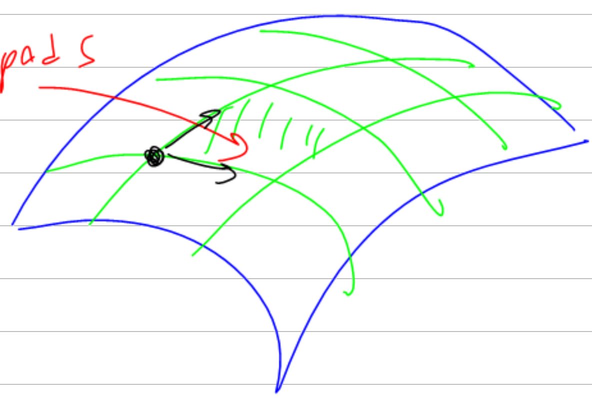


$$\int_{[a,b]} f' = \int_{\partial [a,b]} f = f\Big|_a^b$$

$$\partial[a,b] = \{a,b\}$$



Functions $\rightsquigarrow$ differential forms.



$\overbrace{\text{k-dim subspace}}^{\text{curved}}$ of $\mathbb{R}^n$

A differential form takes a pt in $\mathbb{R}^n$, plus k "tangent vectors" (vectors starting at that pt) & spits out a number.

Linear Algebra: [Fix a f.d. v.s. $V/\mathbb{R}$]

<u>Def</u> A function $T: V^k \longrightarrow \mathbb{R}$ is "multi-linear" if

$$T(v_1, v_2, \ldots, av_j' + bv_j'', \ldots v_k) =$$

$$aT(v_1, \ldots, v_j', \ldots v_k) + bT(v_1 \ldots v_j'', \ldots v_k)$$

(Follows $T(v_1 \ldots 0 \ldots v_k) = 0$)

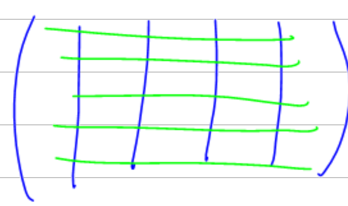
Follows that $T(v_1, \ldots, \lambda v_j, \ldots v_k) = \lambda T(\ldots)$

<u>Examples</u> 1. inner products

$$\langle\, , \rangle : V^2 \longrightarrow \mathbb{R}$$

2. $\det : M_{n\times n} \longrightarrow \mathbb{R}$

$$M_{n\times n} = (\mathbb{R}^n)^n \longrightarrow \mathbb{R}$$



is multi-linear "dets are linear in the col's of the matrix"

$T^k(V) =$ The set of all k-linear things on V = "k-tensors"

Claim $\mathcal{T}^k(V)$ is itself a vector space.

$T_1, T_2 \in \mathcal{T}^k(V)$,

$$(T_1+T_2)(v_1 \dots v_k) = T_1(v_1 \dots v_k) + T_2(v_1 \dots v_k)$$

$$(\underset{\in \mathbb{R}}{a}\ \underset{\in \mathcal{T}^k(V)}{T})(v_1 \dots v_k) = a \cdot (T(v_1 \dots v_k))$$

$\square$

$0 \in \mathcal{T}^k(V) \quad 0(v_1 \dots v_k) = \cancel{2?}\ 0.$

What is a $\underset{k}{\overset{\shortparallel}{0}}$-tensor?

$\boxed{0! = 1}$

Ans is a machine that takes 0 vectors (nothing) and outputs a scalar $\in \mathbb{R}$

Namely, it is a number.

$$\mathcal{T}^0(V) = \mathbb{R}$$

Def if $T_1 \in \mathcal{T}^k(V)$, $T_2 \in \mathcal{T}^l$

then $T_1 \otimes T_2 \in \mathcal{T}^{k+l}$ is defined by
tensor product

$$(T_1 \otimes T_2)(v_1 \ldots v_k \mid v_{k+1} \ldots v_{k+l}) := T_1(v_1 \ldots v_k) \cdot T_2(v_{k+1} \ldots v_{k+l})$$

This really is in $\mathcal{T}^{k+l}$

(~~Claim~~ $\cancel{\bigoplus_{k \geq 0} \mathcal{T}^k(V)}$ ~~is a~~ ~~alg.~~ ~~algebra with unit.~~)

Namely 1. $(T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3$ in $\mathcal{T}^{k+l}$.
($T_1 \in \mathcal{T}^k$, $T_2 \in \mathcal{T}^k$, $T_3 \in \mathcal{T}^l$)

2. $T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3$

3. $(T_1 \otimes T_2) \otimes T_3 = T_1 \otimes (T_2 \otimes T_3)$ as $k+l+m$ tensors.
($T_1 \in \mathcal{T}^k$, $T_2 \in \mathcal{T}^l$, $T_3 \in \mathcal{T}^m$)

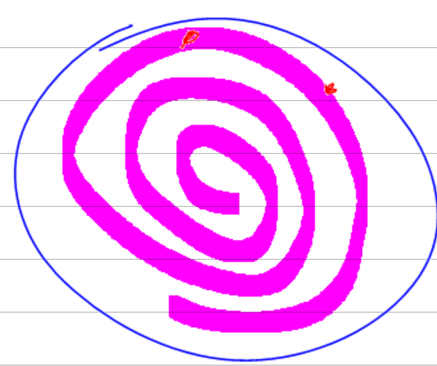
4. $(\alpha T_1) \otimes (\beta T_2) = (\alpha\beta)(T_1 \otimes T_2)$

Exercise What is $\mathcal{T}^0$?

$$\begin{array}{ccccc} U & W & \longmapsto & & U \otimes W \\ \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & & & \rotatebox{90}{$\in$} \\ u & w & \longmapsto & & u \otimes w \end{array}$$

$$\begin{array}{ccccc} T^k(V) & \otimes & T^\ell(V) & \cong & T^{k+\ell}(V) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathcal{T}_1 & & \mathcal{T}_2 & \longrightarrow & \end{array}$$

"tensor prods of elements"



MAT257 Analysis II on January 20, 2021.
Read Along: Spivak 75-85. Comment on TT2.
Please be gentle with zoom chat!
* Class-related only.
* Try not to reach the point where the chat is a distraction.
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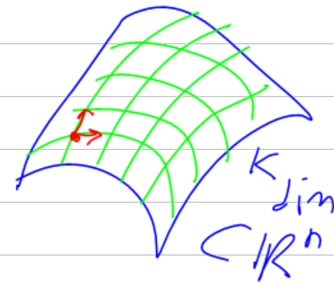
Riddle Along: Let f be a distance-non-inreasing function from the plane to the plane (d(x,y) >= d(f(x),f(y)), for all x,y), and let R be a rectangle in the plane. Is it always true that the length of the boundary of R is greater or equal to the length of the boundary of f(R)?

Remember,

$$\int_C d\omega = \int_{\partial C} \omega$$



warning: Elsewhere $\mathcal{T}^k(V^*)$

Recall $\mathcal{T}^k(V)$ = "k-tensors on V" = $\left\{\begin{array}{c}\text{k-linear maps} \\ V^k \to \mathbb{R}\end{array}\right\}$

a v.s.!

Also, $\mathcal{T}^k \times \mathcal{T}^l \to \mathcal{T}^{k+l}$ via $(T_1, T_2) \mapsto T_1 \otimes T_2$

$$(T_1 \otimes T_2)(v_1 \ldots v_k, v_{k+1}, \ldots v_{k+l}) = T_1(v_1 \ldots v_k) \cdot T_2(v_{k+1} \ldots v_l)$$

Not commutative

$$(T_2 \otimes T_1)(v_1 \ldots v_{k+l}) = T_2(v_1 \ldots v_l) \cdot T_1(v_{l+1}, \ldots v_{k+l})$$

<u>Example</u> $\mathcal{T}^1(V) = \left\{\begin{array}{c}\text{linear functionals} \\ \text{on } V\end{array}\right\}$

$$= \left\{\begin{array}{c}\text{Linear transformation} \\ \varphi: V \to \mathbb{R}\end{array}\right\}$$

$= V^*$ = The dual space of V.

$$\mathbb{R}^n = \left\{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\right\} \ni x \qquad (\mathbb{R}^n)^* = \underline{\underline{\text{Sn}}} = \{(b_1 \ldots b_n)\} \ni \varphi$$

$$\varphi(x) = (b_1 \ldots b_n)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum b_i a_i$$

If V has basis $\beta=(v_1,\dots,v_n)$

then there is a unique basis $\beta^*=(\varphi_1\dots\varphi_n)$ of V^*

w/ the property that

$$\varphi_i(v_j)=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}=\delta_{ij}$$

Example

$$\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right)^*=\big((1\ \ 0),(0\ \ 1)\big)$$

$$\left(\underset{v_1}{\begin{pmatrix}1\\0\end{pmatrix}},\underset{v_2}{\begin{pmatrix}1\\1\end{pmatrix}}\right)=\left(\underset{\varphi_1}{(\overset{1}{a}\ \ \overset{-1}{b})},\underset{\varphi_2}{(\overset{0}{c}\ \ \overset{1}{d})}\right)$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\varphi_1(v_1) & \varphi_1(v_2)\\ \varphi_2(v_1) & \varphi_2(v_2)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$$

The dual basis for the basis formed by the columns of a matrix A is formed by the rows of A^{-1}.

$T^k(V)$ is it f.d? what is $\dim T^k(V)$?

<u>Thm</u> Suppose V is a f.d. v.s. w/ basis $v_1 \ldots v_n$, let $\varphi_1 \ldots \varphi_n$ be the dual basis (of V^*) (meaning $\varphi_i(v_j) = \delta_{ij}$). Then

$$\{\varphi_{i_1} \otimes \varphi_{i_2} \otimes \varphi_{i_3} \otimes \ldots \otimes \varphi_{i_k} : \forall \alpha\ 1 \le i_\alpha \le n\}$$

is an (unordered) basis of $T^k(V)$

Hence $\dim T^k(V) = n^k$.

<u>Pf</u> ~~Lemma~~ Any k-tensor T is determined by its values on basis vectors: $w_i = \sum_{j=1}^{n} a_{ij} v_j$

$$T(w_1 \ldots w_k) = T\left(\sum_{j_1=1}^{n} a_{1j_1} v_{j_1}, \sum_{j_2=1}^{n} a_{2j_2} v_{j_2}, \ldots\right)$$

$$= \sum_{j_1, j_2, \ldots j_k = 1}^{n} a_{1j_1} a_{2j_2} \ldots a_{kj_k}\, T(v_{j_1}, v_{j_2} \ldots v_{j_k})$$

So if $T_1 = T_2$ on basis vects

$\Rightarrow T_1 = T_2$ on all vects.

$^*(\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_k})(v_{j_1}, \ldots, v_{j_k}) = \varphi_{i_1}(v_{j_1}) \ldots \varphi_{i_k}(v_{j_k}) = \delta_{i_1 j_1} \delta_{i_2 j_2} \ldots \delta_{i_k j_k}$

$(i_1 \ldots i_k) = \bar{I} \qquad \bar{J} = (j_1 \ldots j_k)$

$$\varphi_{\bar I}(v_{\bar J}) = \delta_{\bar I \bar J} = \begin{cases} 1 & \bar I = \bar J \\ 0 & \text{otherwise} \end{cases}$$

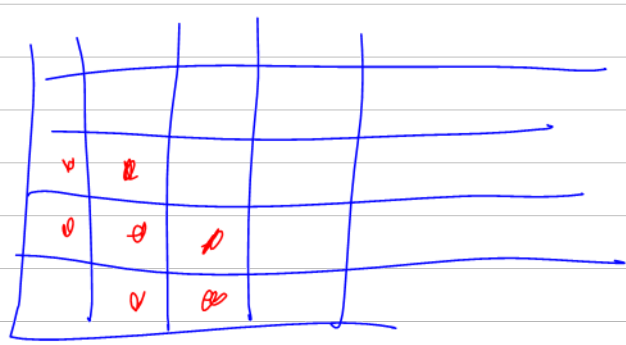
<u>Span</u> Suppose $T \in \mathcal{T}^k(V)$

$$T = \sum a_{\bar I} \varphi_{\bar I}$$

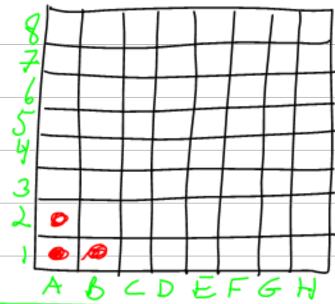
$a_{\bar I} = T(v_{\bar I})$. Indeed

$$T(v_{\bar J}) = a_{\bar J}$$

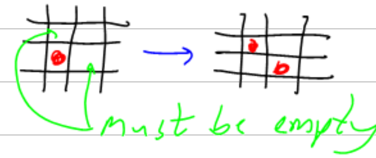
$$\left(\sum a_{\bar I} \varphi_{\bar I}\right)(v_{\bar J}) = \sum_{\bar I} a_{\bar I} \delta_{\bar I \bar J} = a_{\bar J} \qquad \square$$



MAT257 Analysis II on January 22, 2021: Alternating Tensors.
HW11, due next Wednesday, is on the web!
Read Along: Spivak 75-85.
Riddle Along: On a chessboard, there are three pawns at the lower left (at A1, A2, and B1). On each move, pick up one pawn, remove it and place one new pawn to the right and one new pawn above, but only if these squares are unoccupied.



A move:



Notation $\underline{n}=\{1,\dots,n\}$ $\quad J=I\in\underline{n}^k$ means $I=J=(j_1,\dots,j_k)$

If $v_j\in V$, $\quad V_I=(v_{j_1}\dots v_{j_k})\in V^k$ $\quad |\underline{n}^k|=n^k$

If $\varphi_j\in V^*$, $\quad \varphi_I=\varphi_{j_1}\otimes\dots\otimes\varphi_{j_k}\in\mathcal{T}^k(V)$ a tensor

Thm V w/ basis $v_1\dots v_n$; $\varphi_1\dots\varphi_n$ the dual basis

Then $\{\varphi_I:I\in\underline{n}^k\}$ is a basis of $\mathcal{T}^k(V)$.

Hence $\dim\mathcal{T}^k(V)=n^k$.

Done: 1. $T_1=T_2$ in $\mathcal{T}^k(V)$ iff $\forall I\ T_1(V_I)=T_2(V_I)$.

2. $\varphi_I(V_J)=\delta_{IJ}$. $\quad \varphi_i(v_j)=\delta_{ij}$

Pf of Thm:
Spans

$$T\overset{?}{=}\sum_{I\in\underline{n}^k}a_I\varphi_I \quad \text{/ eval on arbit. } V_J$$

each $a_I\in\mathbb{R}$

$a_{113}, a_{123}, a_{231},$

$$T(V_J)=\sum_I a_I\varphi_I(V_J)=\sum_I\delta_{IJ}a_I=a_J$$

$\checkmark$

Given $T\in\mathcal{T}^k(V)$ set $a_I=T(V_I)$, and now $T=\sum a_I\varphi_I$. Indeed it is enough to check this on V_J, and then

$$T(V_J) \stackrel{?}{=} \sum_I a_I \varphi_I(V_J) = \sum_I a_I \delta_{IJ} = a_J$$

!

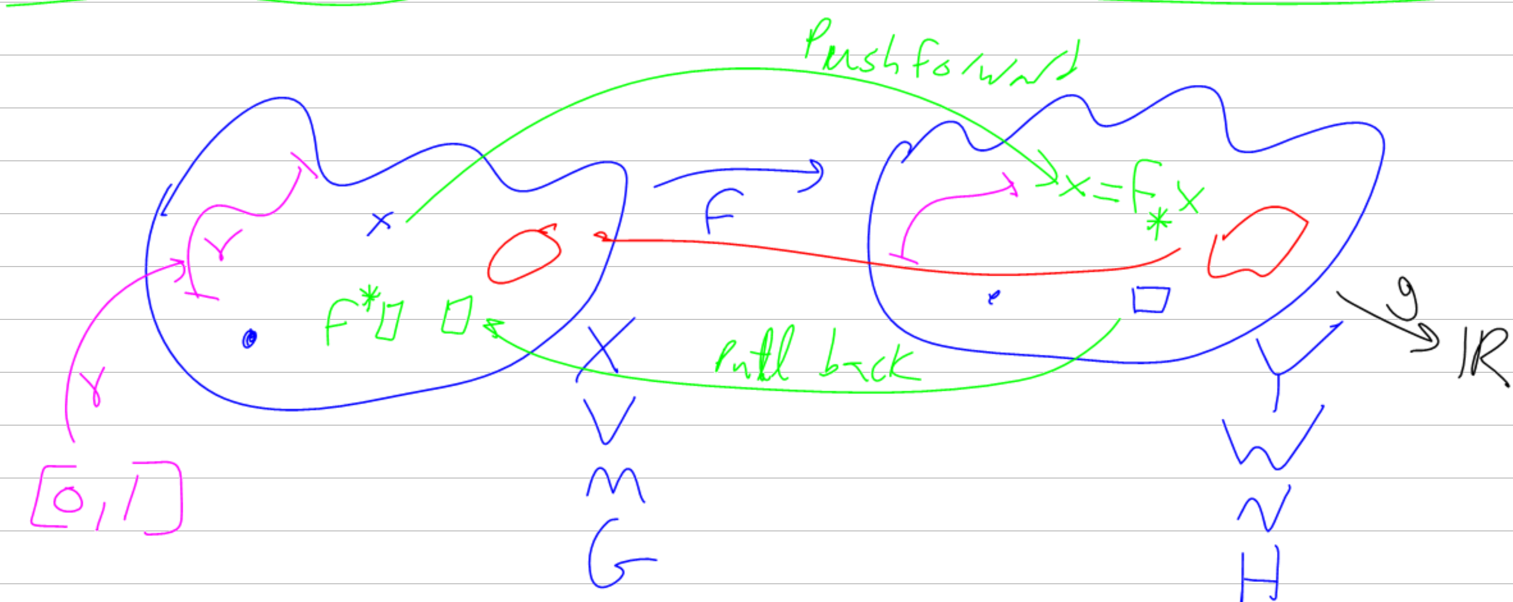
Lin Ind

Suppose $\displaystyle\sum_I a_I \varphi_I = 0$

eval on V_J

$$a_J = \sum_I a_I \varphi_I(V_J) = 0$$

$\Rightarrow a_J = 0$ for all J. $\square$



Points like to be pushed $\quad x \in X \quad F_* x = F(x) \in Y$

paths like to be pushed $\quad F_* \gamma = F \circ \gamma$

Functions like to be pulled $\quad F^* g = g \circ F$

$F(A) \quad F^{-1}(A)$

Subsets like to be pulled: $\quad \underset{A \subset Y}{F^* A} = F^{-1}(A) = \{x : F(x) \in A\}$

$$V \xrightarrow[\text{Linear map}]{F} W \xrightarrow{\varphi} \mathbb{R}$$

$$v \longrightarrow F_* v = F(v)$$

$$F^*\varphi = \varphi \circ F \in V^* \longleftarrow \varphi \in W^*$$

Conclusion: Given $F: V \to W$ there is a "pullback map" $F^*: W^* \to V^*$

$$V \xrightarrow[\text{linear}]{F} W$$

$$T^k(V) \xleftarrow[\text{pullback}]{F^*} T^k(W)$$

Precisely, if $T \in T^k(W)$ then

$$\boxed{\begin{aligned} (F^*T)(u_1, \ldots, u_k) &= T(F_* u_1, \ldots, F_* u_k) \\ \text{where } u_i \in V \quad &= T(F(u_1), \ldots, F(u_k)) \end{aligned}}$$

<u>Claim</u> If $T \in T^k(W)$ then $F^*T \in T^k(V)$, namely, F^*T is multi-linear.

<u>Example</u> Suppose B is an inner prod. on a v.s. V: $B: V \times V \to \mathbb{R}$ s.t.

1. B is bilinear.

2. B is symmetric: $B(x,y) = B(y,x)$

3. B is "pos-definite"

$$B(x,x) \geq 0 \text{ with equality iff } x=0.$$

An inner product is just a 2-tensor hence it likes to be pulled.

Q. When does property 3 fail for f^*B?

The Gram-Schmidt orthogonalization process implies that if (V, B) is a v.s. w/ an inner product, then V has an "orthonormal basis": basis $\{v_i\}$

$$B(v_i, v_j) = \delta_{ij}$$

Given ~~that~~ such a basis, define $f: \mathbb{R}^n \longrightarrow V$ by $f(e_i) = v_i$

$? = f^*B$

$$(f^*B)(e_i, e_j) = B(f_* e_i, f_* e_j) = B(v_i, v_j) = \delta_{ij} = \langle e_i, e_j \rangle$$

$$\Rightarrow f^*B = \langle \, , \, \rangle.$$

Gram-Schmidt $\Rightarrow$ For any (V, B),

$\exists F : \mathbb{R}^n \longrightarrow V$ s.t. $F^*B = \langle \, , \, \rangle$