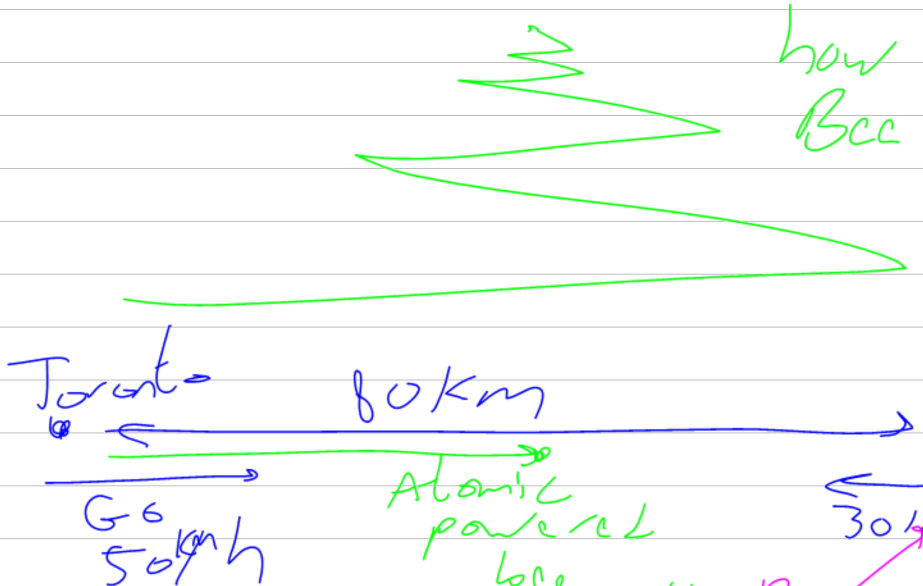
how far does Bee fly before crash?



$|P_A - P_B|$

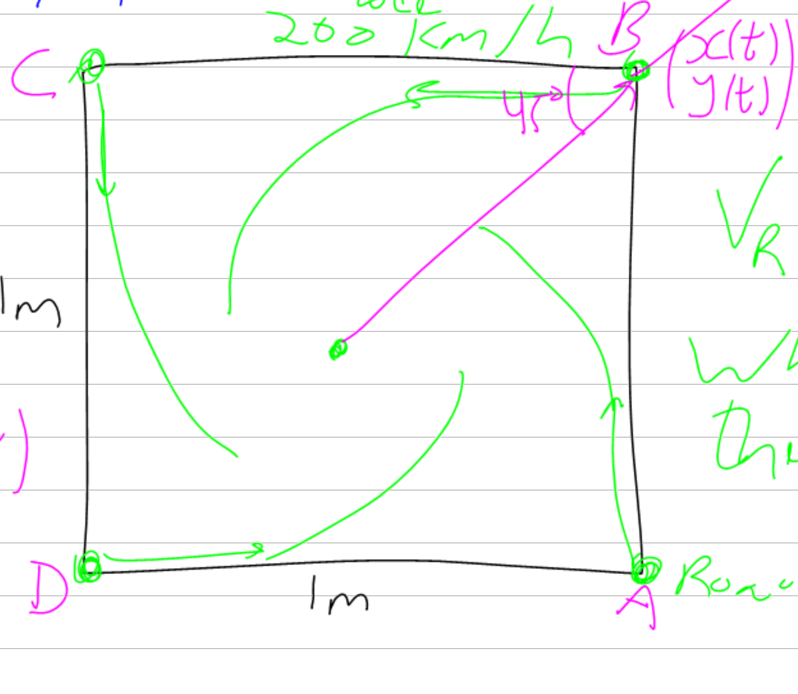
$\|$

$d_{AB}(t)$

$\partial_t d_{AB}(t)$

$\|$

?



$x(0) = +\frac{1}{2} = y(0)$

$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \cos 135 \\ -\sin 135 \end{pmatrix} \frac{\begin{pmatrix} x \\ y \end{pmatrix}}{\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|}$

$V_R = 1m/s$

When will they meet?

MAT257 Analysis II on January 11, 2021: Proof of the COV formula (3).

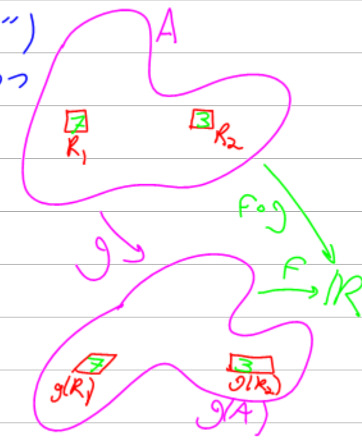
Read Along: Spivak 66-74.

TT2 next week, discussion Wednesday.

COV Strategy: Show that every g is a composition of layer-preserving maps, and use dimensional reduction on those.

Where we were:

Thm (Change of Variables, "COV") Let $A\subset\mathbb{R}^n$ be open, $g:A\to\mathbb{R}^n$ cont. diffable, 1-1, and s.t. $\forall x\in A$ $g'(x)$ is invertible. If $F:g(A)\to\mathbb{R}$ is integrable, then
$$\int_{g(A)}F=\int_A (F\circ g)|\det g'|$$

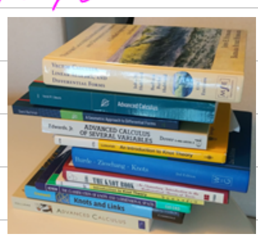


Debts.

1. ~~COV(n-1) ⇒ COV(n) for l.p. maps~~
2. ~~Every g is a composition of l.p. maps & coord swaps.~~
3. ~~COV(g), COV(h) ⇒ COV(g∘h)~~
4. COV holds for coordinate swaps
5. ~~local COV ⇒ global COV~~
6. ~~Prove COV(1)!~~
7. COV(cont) ⇒ COV(integ)

Layer preserving maps:

$\xrightarrow{l.p.}$ layer preserving



$$g:\mathbb{R}^n_x\to\mathbb{R}^n_y \qquad g\begin{pmatrix}x_1\\ \vdots\\ x_{n-1}\\ x_n\end{pmatrix}=\begin{pmatrix}g_1(x)\\ \vdots\\ g_{n-1}(x)\\ x_n\end{pmatrix}$$
$$g_n(x)=x_n$$

COV(n-1) ⇒ COV(n)

Lemma 1 Assume COV(n-1). Let $U\subset\mathbb{R}^n$ be open & bndd. Let $g:U\to\mathbb{R}^n$ be a l.p. map cont. diffable, 1-1, $g'(x)$ invertible & $g(U)$ is bndd. Then a restricted COV holds. Namely, if $F:g(U)\to\mathbb{R}$ is bndd & cont. & $\operatorname{supp}F\subset g(U)$, then
$$\int F=\int (F\circ g)|\det g'|$$

Proof Use Fubini on both sides.

Lemma 6. COV(1).

Pf. Follows from the Fundamental Thm of Calculus

Lemma 3 COV(g), COV(h) ⇒ COV(g∘h)

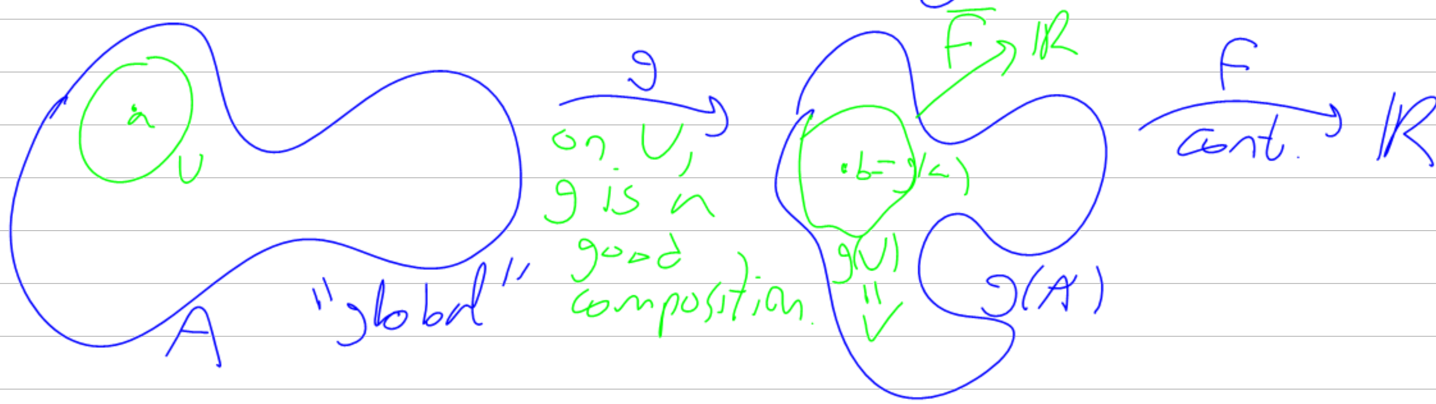
Pf
$$\int_{g(h(A))}F \overset{COV(g)}{=} \int_{h(A)}\underbrace{(F\circ g)|\det g'|}_{\tilde F} \overset{COV(h)}{=} \int_A (\tilde F\circ h)|\det(h')|$$
$$=\int_A (F\circ g\circ h)\,|\det(g'\circ h)|\,|\det(h')|$$
$$=\int_A F\circ(g\circ h)\cdot|\det(g'\circ h)\cdot h'| = \int_A F\circ(g\circ h)\cdot|\det(g\circ h)'|\quad\square$$

Lemma 2 In the conditions of the main thm, For every $a\in A$ there is some open $U\ni a$, s.t. on U g is a composition of coord. swaps & l.p. maps.

Def
$T_{ij}:\mathbb{R}^n\to\mathbb{R}^n$
$$T_{ij}\begin{pmatrix}x_1\\ \vdots\\ x_i\\ \vdots\\ x_j\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}x_1\\ \vdots\\ x_j\\ \vdots\\ x_i\\ \vdots\\ x_n\end{pmatrix}$$

Pf
$g:\mathbb{R}^n_x\to\mathbb{R}^n_y \quad y_i=g_i(x_1\dots x_n)$
$$\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\overset{\alpha_k}{\longmapsto}\begin{pmatrix}x_1\\ \vdots\\ x_{n-1}\\ y_k\end{pmatrix}\overset{\beta_k}{\longmapsto}\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}$$
g
$g_k(x)$

Lemma 5 Local cov $\Rightarrow$ global



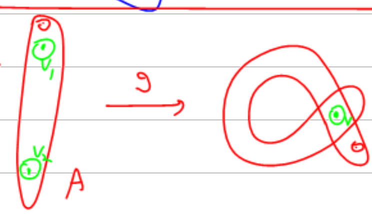
NTS $$\int_A F\circ g \cdot |\det g'| = \int_{g(A)} F$$

Pf POU to the rescue!

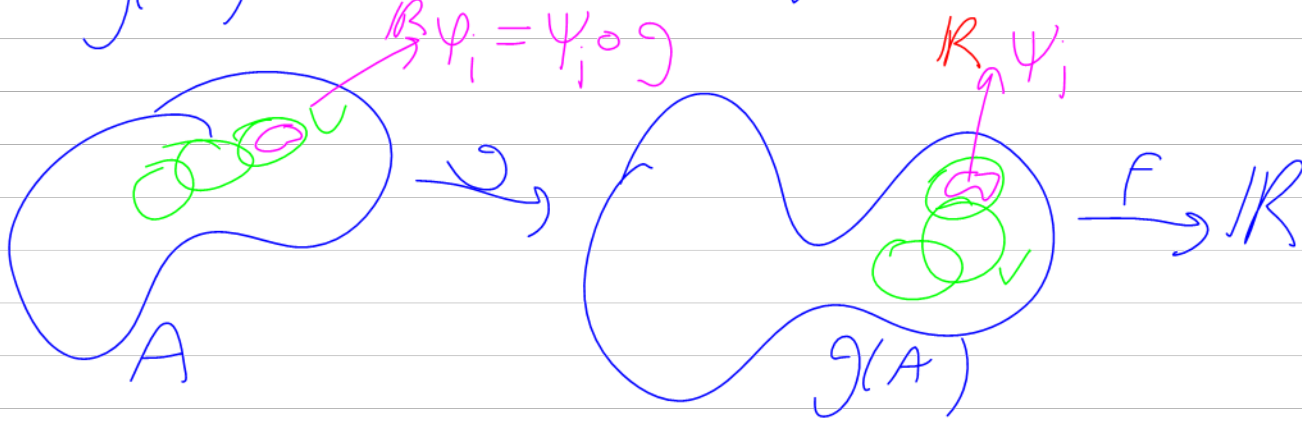
Find a cover $\mathcal{V}$ of $g(A)$ by bndd open set V s.t. $\forall V \in \mathcal{V}$,

$g^{-1}(V)$ is bndd & on it g is a "good" composition.

Aside
what goes wrong if g isn't 1-1



Let $\{\psi_i\}$ be a POU for $g(A)$ subordinated to the cover $\mathcal{V}$



Then $\varphi_i = \psi_i \circ g$ make a PO1 for

A subordinate $\mathcal{U} = \{U = g^{-1}(V)\}_{V \in)}$

Now

$$\int_{g(A)} f = \sum_i \int_{g(A)} f \cdot \psi_i =$$

$$= \sum_i \int_{\cancel{V(\psi_i)}} f \cdot \psi_i \overset{\text{loc}}{\underset{\text{cov.}}{=}} \sum_i \int \big((f \cdot \psi_i) \circ g\big) \cdot |\det g'|$$

$$= \sum_i \int (f \circ g) \overbrace{(\psi_i \circ g)}^{\varphi_i} \, |\det g'|$$

$$= \sum_i \int \varphi_i \, \underline{(f \circ g)} \, \underline{|\det g'|} = \int (f \circ g) |\det g'|$$

$\square$

Office Hours and Term Test 2

Dror Bar-Natan

All Sections

Dear All,

Due to clashes of my current office hours schedule with MAT267, as of next week my office hours will take place on Tuesdays at 9-10 (as before) and at 12-1 (replacing 1-2).

Here's what I'll say in class tomorrow about Term Test 2.

- The test will take place on Tuesday January 19, 5-7PM (Toronto time), on Crowdmark (you will get a link by email about one minute before the official starting time). Other than documented accessibility matters, no exceptions!
- Our TAs Sebastian and Shuyang will hold extra pre-test office hours, in their usual zoom rooms. Sebastian on Monday 11-2 at **Sebastian's Zoom** (https://us02web.zoom.us/j/88310099689?pwd=S1dTNmJDZkRBNW5QWENWcWtyWm5QQT09) (password vchat), and Shuyang on Friday and on Tuesday at 10:30-11:30 at **Shuyang's Zoom** (https://utoronto.zoom.us/j/83428997680) (password vchat). These office hours replace some of their regular office hours; so Sebastian will not hold his regular office hours on January 25 and on February 1, and Shuyang will not hold her regular office hours on January 20 and 27.
- I will hold my regular office hours on Tuesday at 9-10 and 12-1, at **http://drorbn.net/vchat** (http://drorbn.net/vchat).
- I will be available to answer questions throughout the exam, at my usual office (**http://drorbn.net/vchat** (http://drorbn.net/vchat), but I'll add a waiting room). I will also be monitoring my regular email address (**drorbn@math.toronto.edu** (mailto:drorbn@math.toronto.edu)) throughout the exam.
- There will be mishaps! I just hope that not too many. If you encounter one, document everything with specific details, times, and screen shots, and send me a message by Wednesday January 20 at 7PM. I will deal with these situations on a case by case basis.
- Don't let unanswered questions and/or mishaps paralyze you! If you need an answer but for whatever reason you cannot reach me, think hard, come up with what you think is the most reasonable answer/resolution, document as best as you can (for example, by adding a note on your submission), and act following your conclusions.
- Material: Everything up to and including the Change of Variables Theorem, with greater emphasis on the material that was not included in Term Test 1 (meaning, starting with integration).
- Open book(s) and open notes but you can only use the internet (during the exam) to read the exam, to submit the exam, and to connect with the instructor/TAs to ask clarification questions. No contact allowed with other students or with any external advisors, online or in person.
- The format will be "Solve 7 of 7", or maybe "6 of 6" or "5 of 5".
- You will be required to copy in your handwriting and sign an academic integrity statement and submit it on Crowdmark along with the rest of your exam. If you wish, you may save time by preparing the academic integrity statement in advance as in **this sample** (http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT1-LastTask.png).
- You will be given an extra 20 minutes at the end of the exam to upload it and to copy/sign the academic integrity statement.
- The vast majority of students will do honest work, and I appreciate that. Out of respect for the honest students I will do my best to pursue and punish any cheating that may occur. I'm more experienced than you! If you plan to be dishonest, think again.
- To prepare: Do the TT2 "rejects" available **here** (http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT2-Rejects.pdf), but more important: make sure that you understand every single bit of class material so far!
- It is not the test I want! Class material and HW are important, but there won't be questions straight from class/HW. Many things in 2020/21 are not as we want them.

Best,

Dror.

MAT257 Analysis II on January 13, 2021:
Proof of the COV formula (4) and Baby Sard.

Read Along: Spivak 66-74.

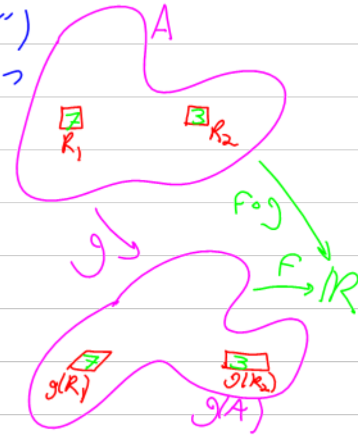
TT2 next week on Tuesday, 5-7PM.

COV Strategy: Show that every g is a composition of layer-preserving maps, and use dimensional reduction on those.

Where we were:

Thm (Change of Variables, "COV") Let $A\subset\mathbb{R}^n$ be open, $g: A\to\mathbb{R}^n$ cont. diff'able, 1-1, and s.t. $\forall x\in A$ $g'(x)$ is invertible. If $f: g(A)\to\mathbb{R}$ is integrable, then

$$\int_{g(A)} f = \int_A (f\circ g)|\det g'|$$

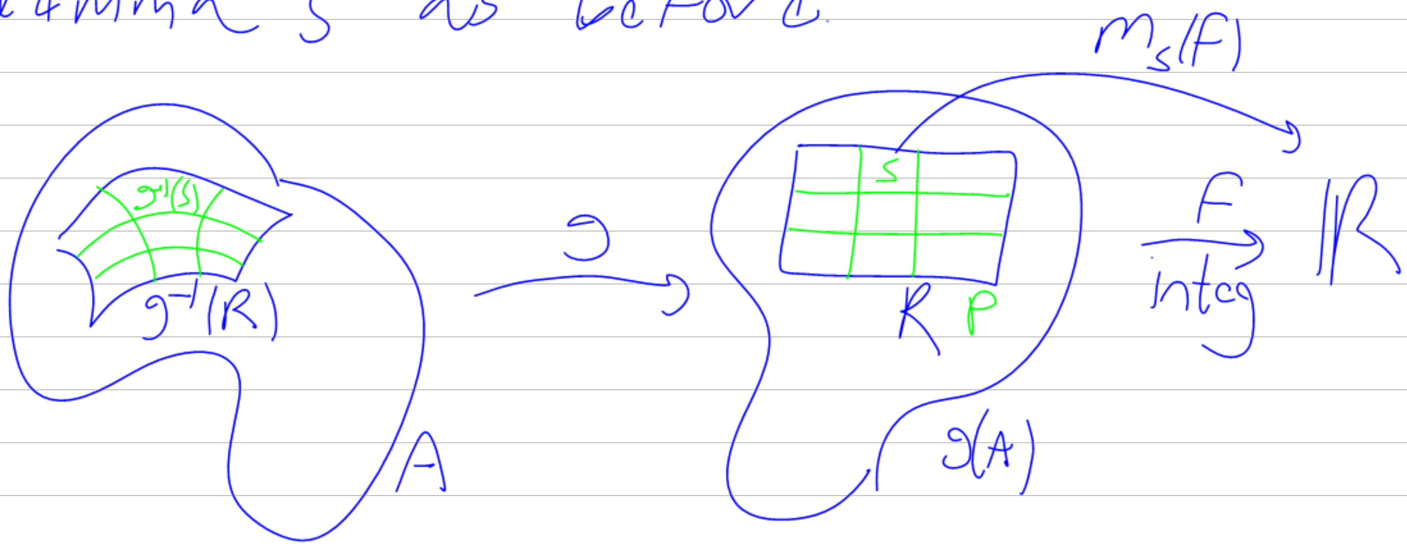


Debts.

1. ~~COV(n-1) ⇒ COV(n) for l.p. maps~~
2. ~~Every g is a composition of l.p. maps & coord swaps.~~
3. ~~COV(g), COV(h) ⇒ COV(g∘h).~~
4. COV holds for coordinate swaps
5. ~~local COV ⇒ global COV.~~
6. ~~Prove COV(1)!~~
7. ~~COV(cont) ⇒ COV(integ)~~

Lemma 7 Suppose COV holds for ~~cont.~~ const. functions f, Then it also holds for arbitrary integrable f.

Pf We'll prove a local version, this version can be made global using lemma 5 as before.



NTS: $$\int_R f = \int_{g^{-1}(R)} (f\circ g)\,|\det g'|$$

$$\inf_{x\in S} f(x)$$

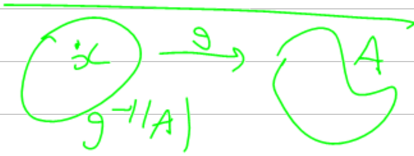
$$L(f,P) = \sum_{S\in P} v(S)\cdot m_S(f)$$

$f\circ g$ on $g^{-1}(S)$ $\geq$

$$= \sum_{S\in P} \int_S m_S(f) = \sum_S \int_{g^{-1}(S)} m_S(f)\cdot|\det g'|$$

$$\leq \sum_{S\in P} \underline{\int}_{g^{-1}(S)} (f\circ g)\cdot|\det g'|$$

$$= \sum_{S\in P} \underline{\int}_{g^{-1}(R)} \overset{(\chi_S\circ g)}{\overset{\|}{\chi_{g^{-1}(S)}}}\cdot(f\circ g)\cdot|\det g'|$$



$$\chi_{g^{-1}(A)} = \chi_A\circ g$$

$$= \sum_{S\in P} \underline{\int}_{g^{-1}(R)} ((\chi_S\cdot f)\circ g)\,|\det g'|$$

Lemma 7'

$$\underline{\int} h_1 + \underline{\int} h_2 \leq \underline{\int} h_1 + h_2$$

$$\cancel{=}\ \leq \underline{\int}_{g^{-1}(R)} \Big(\underbrace{\big(\sum_S \chi_S\big)}_{\chi_R\circ g\,\equiv\, 1 \text{ on } g^{-1}(R)} f\Big)\circ g \cdot |\det(g')|$$

$$= \underline{\int}(f\circ g)\cdot|\det(g')| \le \overline{\int} f\circ g\,|\det g'| \le U(f,P)$$

So

$$L(f,P) \le \underline{\int} f\circ g\,|\det g'| \le \overline{\int} (f\circ g)\cdot|\det g'| \le U(f,P)$$

differ by $<\epsilon$

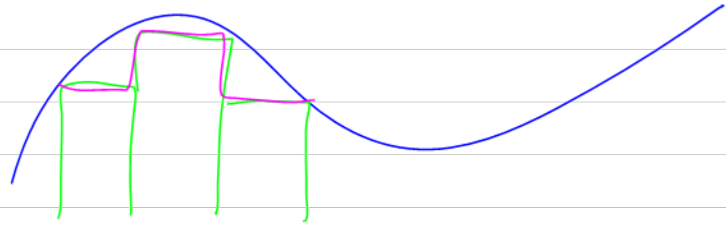
differ by $<\epsilon$

So $\underline{\int} = \overline{\int}$ so $\int$ exists, and is sandwiched between $L(f,P)$ & $U(f,P)$ so $\int f\circ g\,|\det g'|$ is $\int_R f$

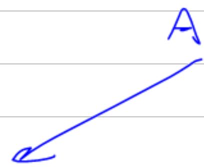
Claim $\underline{\int}_R h_1 + \underline{\int}_R h_2 \le \underline{\int}_R h_1 + h_2$

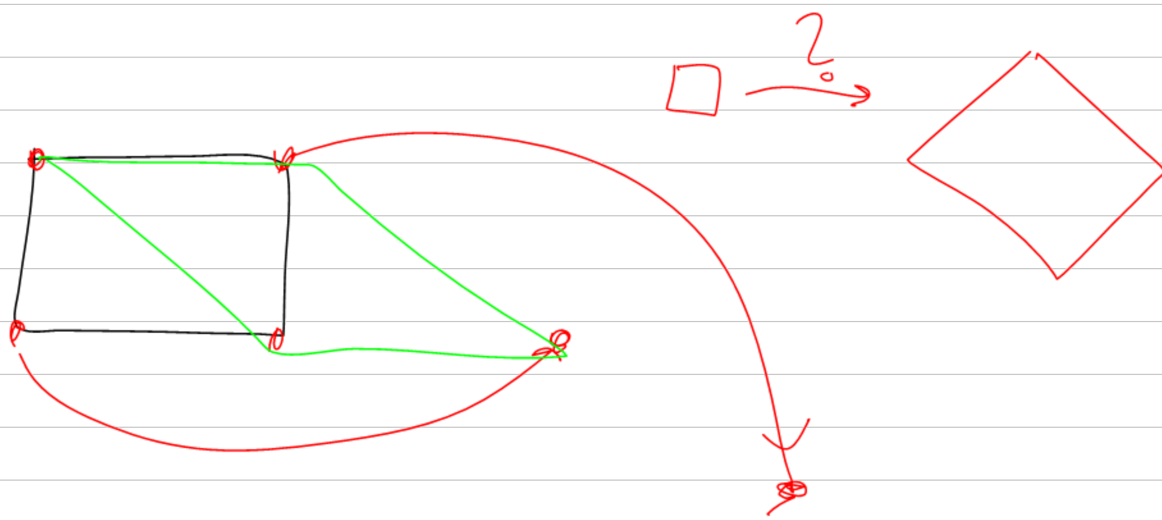
$$L(h_1+h_2,P) = \sum_{S\in P} v(S)\, \overbrace{m_S(h_1+h_2)}^{\geq\; m_S(h_1)+m_S(h_2)}$$

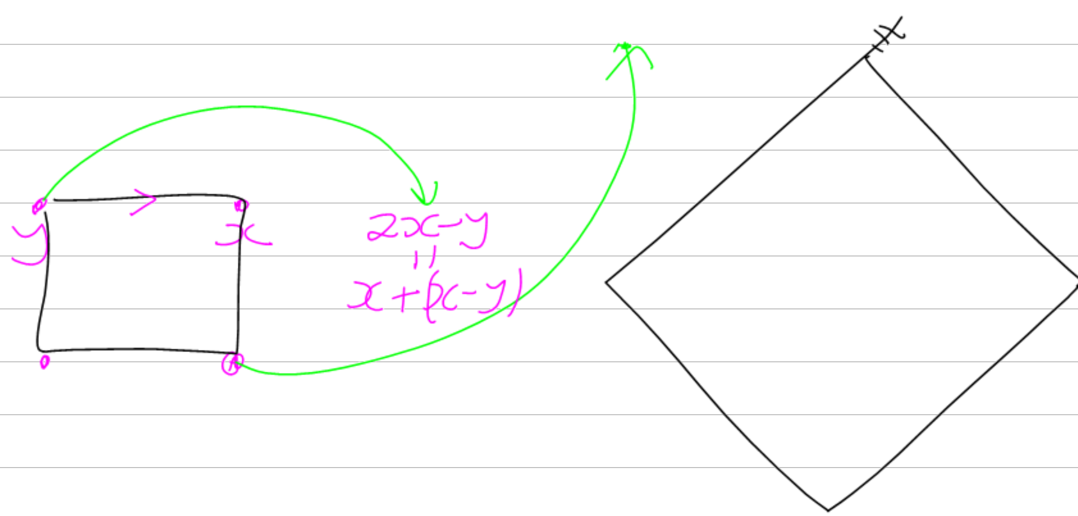
$$\ge \sum_S v(S)\left[m_S(h_1) + m_S(h_2)\right] = L(h_1,P) + L(h_2,P) \quad \square$$



$$\int = \int$$







MAT257 Analysis II on January 15, 2021: Coordinate swaps and Baby Sard.
Read Along: Spivak 66-74.
TT2 next week on Tuesday, 5-7PM.
Riddle Along: Four points form a perfect square in the plane. Can you turn that square into a larger one by a sequence of moves of the form (x,y)->(x,2x-y) performed on pairs from within our four points? (Namely, by reflections of y about x?)

Leftover:

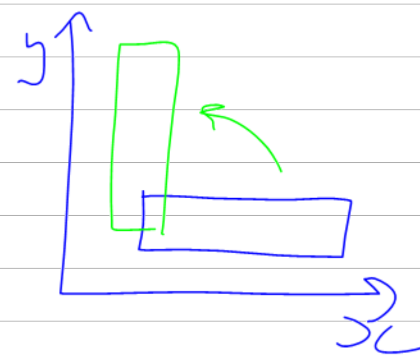
4. cov holds for coordinate swaps

$$T_{ij}: \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_j \\ x_i \\ x_n \end{pmatrix} \qquad T_{ij}' = T_{ij} \qquad |\det T_{ij}'| = |-1| = 1$$

$$\int_{T_{ij}A} f = \int_A f \circ T_{ij} \qquad T: \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto (y,x)$$

Replace f w/ $f\circ T$

NTS:
$$\int_{TA} f\circ T = \int_A f\circ T\circ T = \int_A f$$



$$A = [a_1, b_1] \times [a_2, b_2] \qquad TA = [a_2, b_2] \times [a_1, b_1]$$

$$P = (a_1 = t_{10} \le t_{11} \le \dots \le t_{1n_1} = b_1,\ a_2 = t_{20} \le \dots \le t_{2n_2} = b_2)$$
$$TP = (t_{20} \le t_{21} \le \dots \le t_{2n_2},\ t_{10} \le \dots \le t_{1n_1})$$

$S \in P \qquad TS \in TP$

Part of TA?

$$\boxed{\int_{TA} f\circ T = \int_A f}$$

$$L(f\circ T, P) = \sum_{S\in P} v(S)\cdot m_S(f\circ T)$$
$$= \sum_{S\in P} v(\underbrace{TS}_{S'})\cdot m_{TS}(f) = \sum_{S'\in TP} v(S') m_{S'}(f)$$

$= L(f, \tau P)$

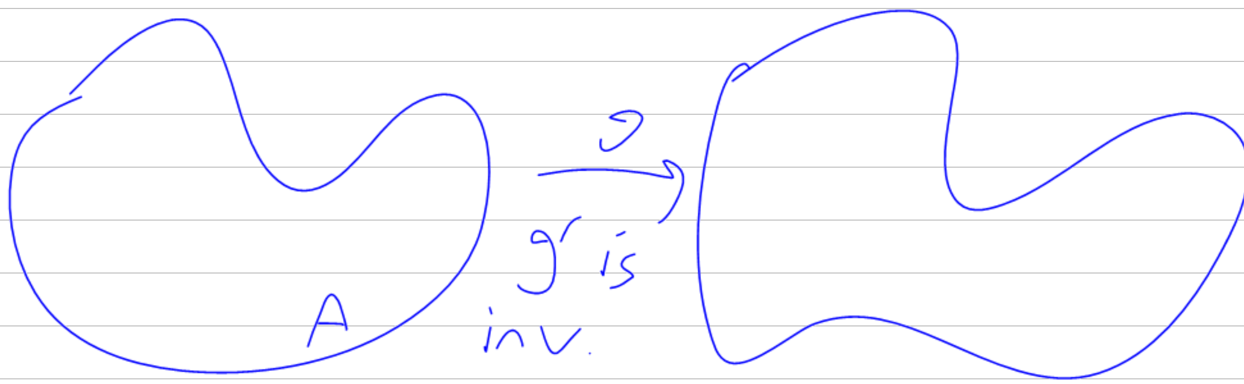
a part. of A.

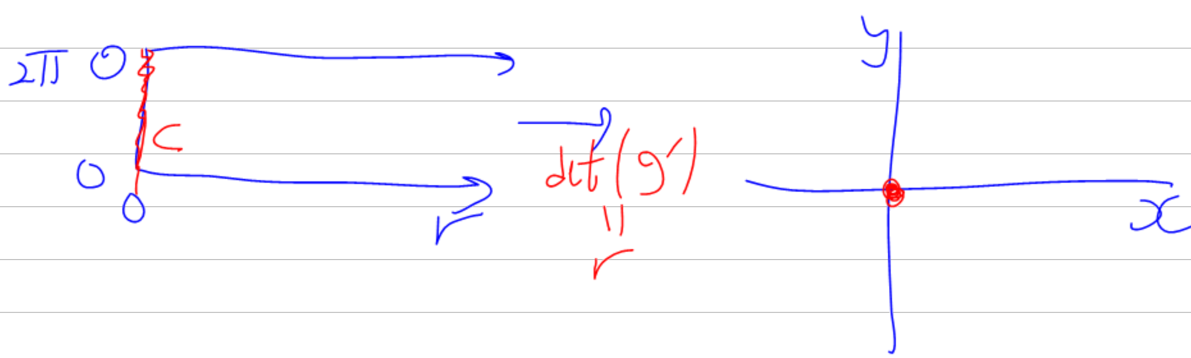
$\boxed{4}$ $\boxed{\text{COV}}$

$$\int_{\tau A} h = \int_A h \circ \tau \qquad \text{Let } f = h\circ\tau$$

$$h = f \circ \tau$$

$$\int_{\tau A} f\circ\tau = \int_A f$$





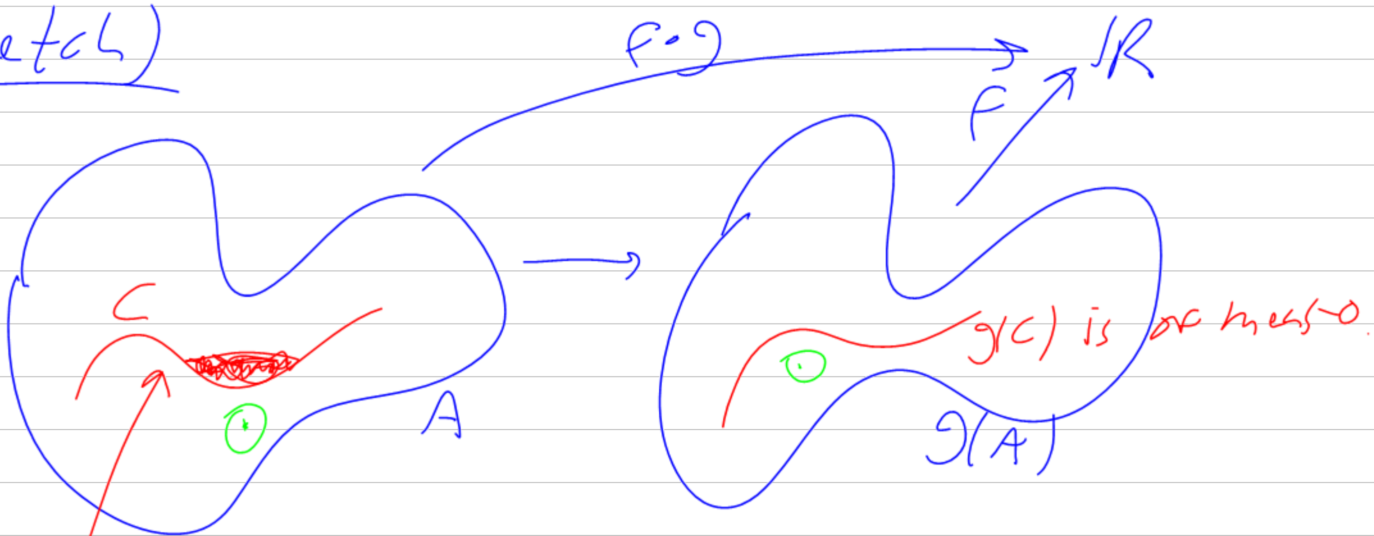
<u>Baby Sard Thm</u> $A \subset \mathbb{R}^n$ open, $g: A \to \mathbb{R}^n$ cont. diff'ble, C = "critical set"

$$C = \{x \in A : \underbrace{\det g'(x)}_{h(x) \text{ is cont.}} = 0\} = h^{-1}(0) \cap A$$

Then $g(C)$ is of meas-0.

<u>~~Comment~~ Corollary 1</u> Can drop the cond. "g' is inv."
from the statement of COV.

<u>Pf (sketch)</u>



C is a closed set. $g(C)$ is also closed.

$$\int_A f\circ g \, |\det g'| \overset{?}{=} \int_{g(A)} f$$

$$\| \qquad\qquad\qquad \|$$

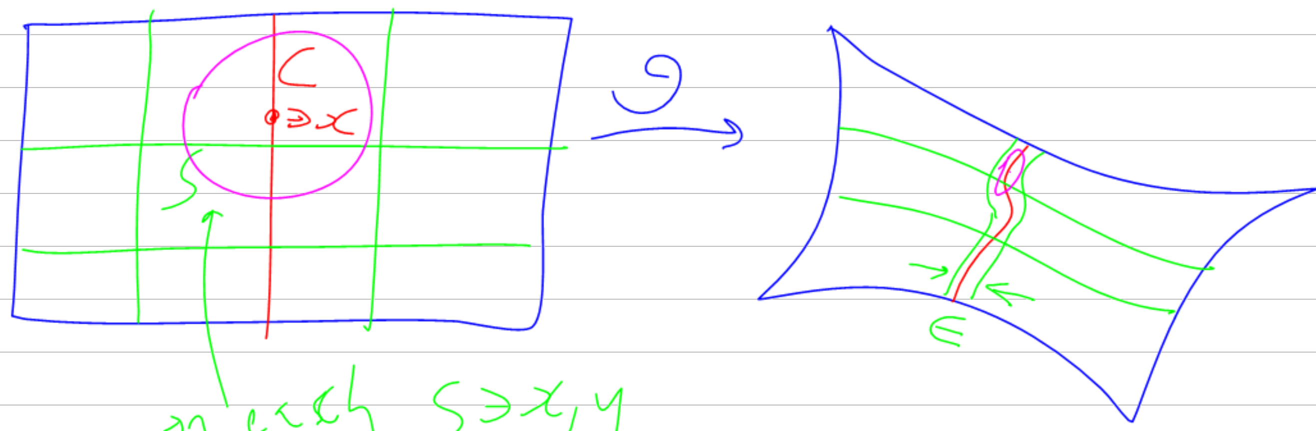
$$\int_{A \setminus C} f\circ g \, |\det g'| \underset{\text{by old COV.}}{\overset{D_o}{=}} \int_{g(A)-g(C)} f$$

<u>Comment 2</u> There is an "adult said" thm.

$$g : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad C = \left\{ x : \operatorname{rank} g'(x) < m \right\}$$

+ some diff'bility cond. $\Longrightarrow$ $g(C)$ is meas-0.

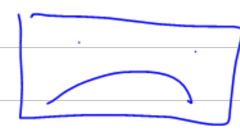
Pf (sketch) of baby Sard. Enough to take A = rect.



on each $S \ni x, y$
$|g'(x) - g'(y)| < \epsilon$

$\Rightarrow$ if $S \in P$, $S \cap C \neq \emptyset \Rightarrow v(g(S)) < \epsilon\, v(S)$

$\Rightarrow \mathrm{Vol}(g(C)) < \text{const} \cdot \epsilon \cdot \mathrm{Vol}(A)$



Riddle: Every open set is a countable ~~union~~ union of open/closed rectangles.