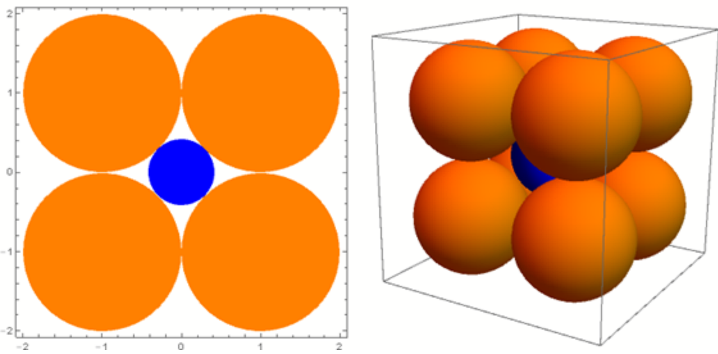
MAT257 Analysis II on December 7, 2020: Volumes of spheres and balls, proof of the COV formula (1).
Reminder: Class on Thursday!
Read Along: Spivak 66-74.

Riddle: Compute lim_{n\to\infty}Vol(B_n)/Vol(C_n), where B_n is the largest ball bounded by 2^n balls of radius ones with centers at {-1,1}^n and C_n is the smallest cubes bounding same balls. Promise: You will learn something very surprising if you solve this riddle.

```
GraphicsRow[{
  Graphics[{Orange, Disk /@ Tuples[{1, -1}, 2], Blue, Disk[{0, 0}, √2 - 1]}, Frame -> True],
  Graphics3D[{Orange, Ball /@ Tuples[{1, -1}, 3], Blue, Ball[{0, 0, 0}, √3 - 1]}]
}, ImageSize → 720]
```



Non-riddle (Cavalieri's principle): Which of the following piles of books weighs the most?



Reminder: "the most important integral in mathematics":

$$I_1 = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

Let's compute like physicists!

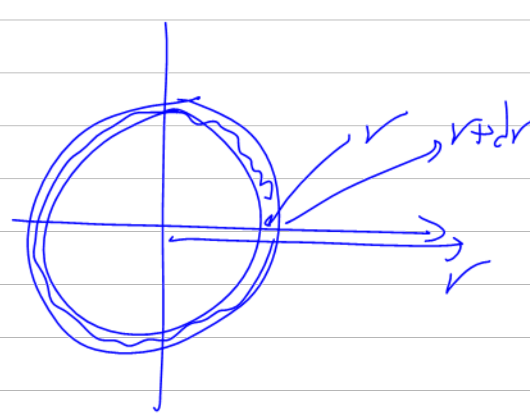
σ_n: "surface area of the n-dimensional sphere" $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$

$S^2 =$ 

$$(2\pi)^{\frac{n+1}{2}}$$

$$\|$$

$$I_1^{n+1} = I_{n+1} = \int_{\mathbb{R}^{n+1}} e^{-|z|^2/2}\,dz = \quad [\|z\| = r]$$



$$= \int_0^{\infty} \underline{dr}\, e^{-r^2/2}\, \underbrace{\mathrm{Vol}(S^n_r)}_{n\text{-dim sense}}$$

$$= \underbrace{\int_0^\infty dr \cdot e^{-r^2/2} \cdot r^n}_{\tau_n} \cdot \sigma_n = \tau_n \cdot \sigma_n$$

$n \geq 2$:

$$\tau_{n-2} = \int_0^\infty dr\, \underbrace{r e^{-r^2/2}}_{D} \underbrace{r^{n-2}}_{I} = -\int (-r)\, e^{-r^2/2} \cdot \frac{r^{n-1}}{n-1}$$

$$= \frac{\tau_n}{n-1}$$

$$\sigma_n = \frac{(2\pi)^{\frac{n+1}{2}}}{\tau_n} = 2\pi \frac{(2\pi)^{\frac{n-1}{2}}}{(n-1)\tau_{n-2}} = \frac{2\pi}{n-1} \sigma_{n-2}$$

$S^0 = \{-1, 1\}$

Examples

$\sigma_0 = 2$ $\qquad \cancel{\beta_0}$

$\sigma_1 = 2\pi$ $\qquad \beta_1 = 2$

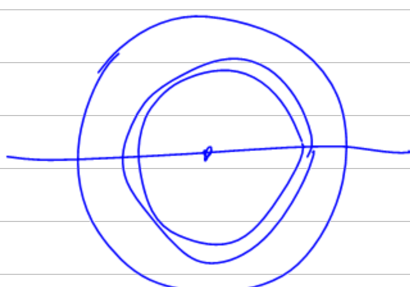
$\sigma_2 = \frac{2\pi}{2-1} \cdot 2 = 4\pi$ $\qquad \beta_2 = \pi$

$\sigma_3 = \frac{2\pi}{3-1} \cdot 2\pi = 2\pi^2$ $\qquad \beta_3 = \frac{4\pi}{3}$

$$\beta_n = Vol(B^n) =$$

$$= Vol\{x \in \mathbb{R}^n : |x| \leq 1\}$$

$$= \int_{B^n} 1$$

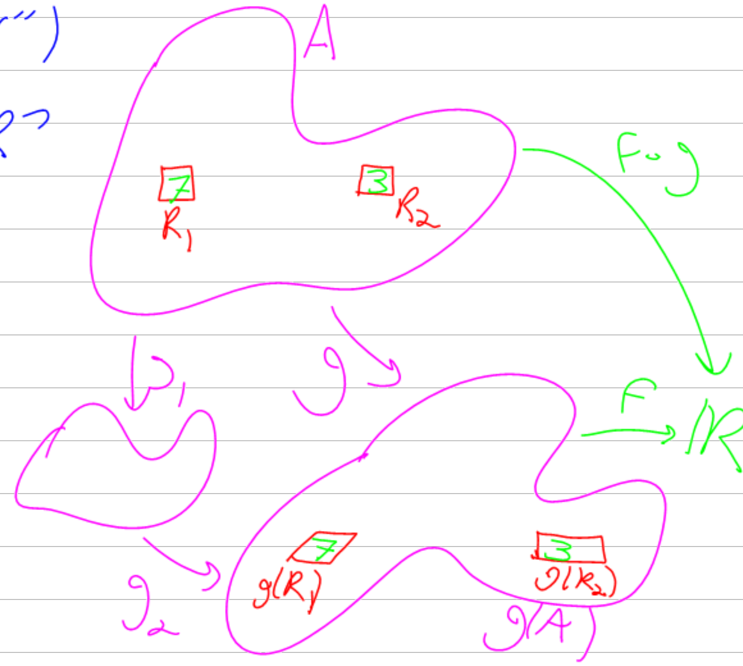


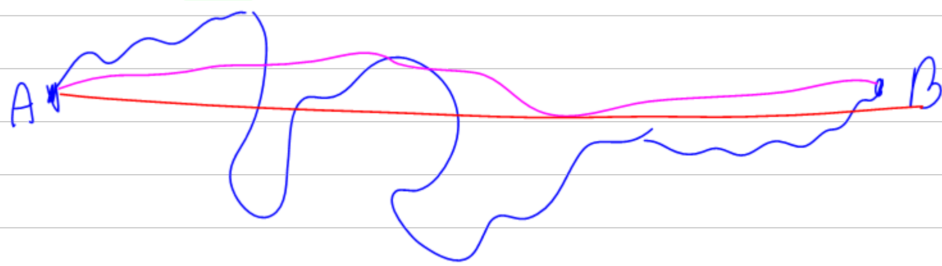
$$= \int_0^1 \sigma_{n-1} r^{n-1} dr = \frac{\sigma_{n-1}}{n}$$

Thm (Change of Variables, "COV")

Let $A \subset \mathbb{R}^n$ be open, $g: A \to \mathbb{R}^n$ cont. diff'able, 1-1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $F: g(A) \to \mathbb{R}$ is integrable, then

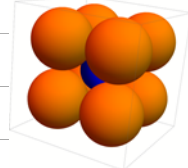
$$\int_{g(A)} F = \int_A (F \circ g)\,|\det g'|$$

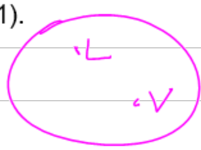




Q: $COV(g_1)$ & $COV(g_2) \Longrightarrow COV(g_1 \circ g_2)$?

MAT257 Analysis II on December 9, 2020:
Proof of the COV formula (1).
Read Along: Spivak 66-74.

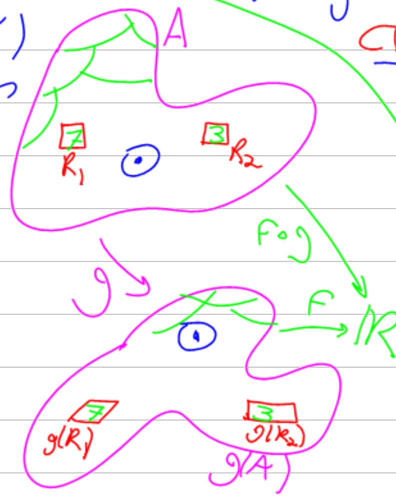




$V_L = V_V$

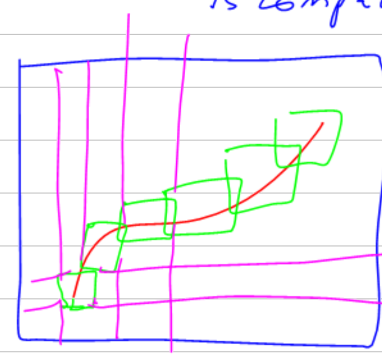
Thm (Change of Variables, "COV") Let $A \subset \mathbb{R}^n$ be open, $g: A \to \mathbb{R}^n$ cont. diff'able, 1-1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $F: g(A) \xrightarrow{} \mathbb{R}$ (open) is integrable, then

$$\int_{g(A)} F = \int_A (F \circ g)|\det g'|$$



Aside Integrals don't change if you modify the integrand/integration domain on a closed set of meas-0.

$\int \to \int \varphi_i F$ supp φ_i is compact.



$\sum \text{vol} < \epsilon$

φ
layer preserving map

$g: \mathbb{R}^n_x \to \mathbb{R}^n_y$ $g\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} g_1(x) \\ \vdots \\ g_{n-1}(x) \\ x_n \end{pmatrix}$

$g_n(x) = x_n$

Debts.

1. $COV(n-1) \Rightarrow COV(n)$ for l.p. maps
2. Every g is a composition of l.p. maps & coord. swaps.
3. $COV(g), COV(h) \Rightarrow COV(g \circ h)$.
4. COV holds for coordinate swaps
5. local COV $\Rightarrow$ global COV.
6. Prove $COV(1)$!

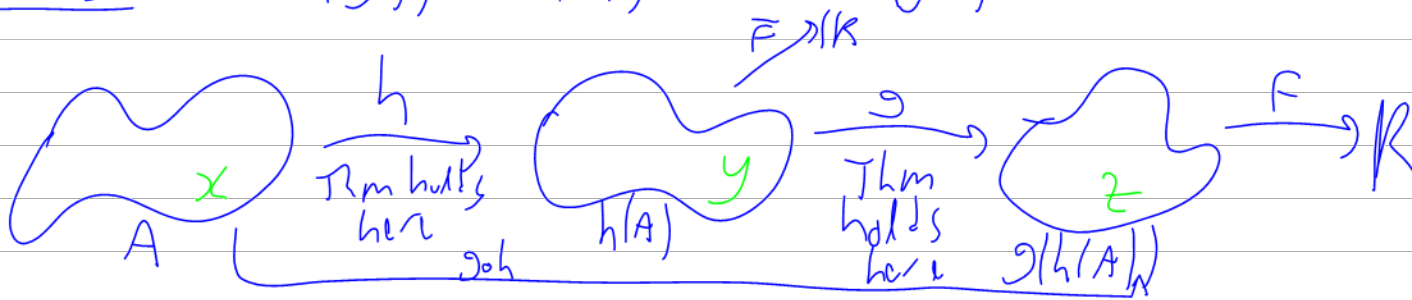
7+ maybe more.

$y_1 = g_1(x_1 \dots x_n)$
$\vdots$
$y_n = g_n(x_1 \dots x_n)$

$\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \leftarrow \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_n \end{pmatrix} \to \begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{pmatrix}$

$\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ g_n \end{pmatrix}$

Lemma 3 $COV(g), COV(h) \Rightarrow COV(g \circ h)$



Pf $\displaystyle\int_{g(h(A))} F \underset{\text{cov}(g)}{=} \int_{y\, h(A)} \underbrace{(F\circ g)\,|\det g'|}_{\bar F}^{F(g(y))\,|\det g'(y)|} \underset{\text{cov}(h)}{=} \int_A (\bar F\circ h)\cdot|\det(h')|$

$$= \int_A (F\circ g\circ h)\cdot|\det(g'\circ h)|\,|\det(h')|$$

$$= \int_A F\circ(g\circ h)\cdot\Big|\det\overbrace{(g'\circ h)\cdot h'}\Big| = \int_A F\circ(g\circ h)\cdot|\det(g\circ h)'| \quad \boxed{\because}$$

Lemma 6 cov hold if $n=1$.

Pf WLOG, $A=(a,b)$. g is 1-1, so g is monotone. So $g(A)=g((a,b))=\begin{cases}(g(a),g(b))\\(g(b),g(a))\end{cases}$

$$\int_{g(A)} F \overset{\text{if } g \text{ is decreasing}}{=} \int_{g(b)}^{g(a)} F = \int_b^a (F\circ g)\,g' = +\int_a^b (F\circ g)\,\underset{\det g'}{\underset{\shortparallel}{|g'|}}$$

$\square$

MAT257 Analysis II on December 8, 2020: Proof of the COV formula (2).
Next class on January 11, nothing until then!
Read Along: Spivak 66-74.
COV Strategy: Show that every g is a composition of layer-preserving maps, and use dimensional reduction on those.

Debts.

1. ~~COV(n-1) ⇒ COV(n) for l.p. maps~~

2. ~~Every g is a composition of l.p. maps & coord swaps~~

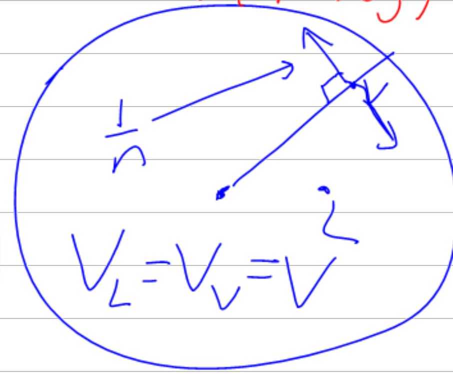
3. ~~COV(g), COV(h) ⇒ COV(g∘h)~~

4. COV holds for coordinate swaps

$\boxed{\text{5. local COV} \Rightarrow \text{global COV}}$

6. ~~Prove COV(1)!~~

7. COV(cont) ⇒ COV(integ)



Lemma 1 Assume COV(n-1). Let $U \subset \mathbb{R}^n$ be open & bndd. Let $g : U \longrightarrow \mathbb{R}^n$ be a l.p. map

$$\left(g\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} g_1(x) \\ \vdots \\ g_{n-1}(x) \\ x_n \end{pmatrix} \text{ or } g_n(x) = x_n \right) \&$$

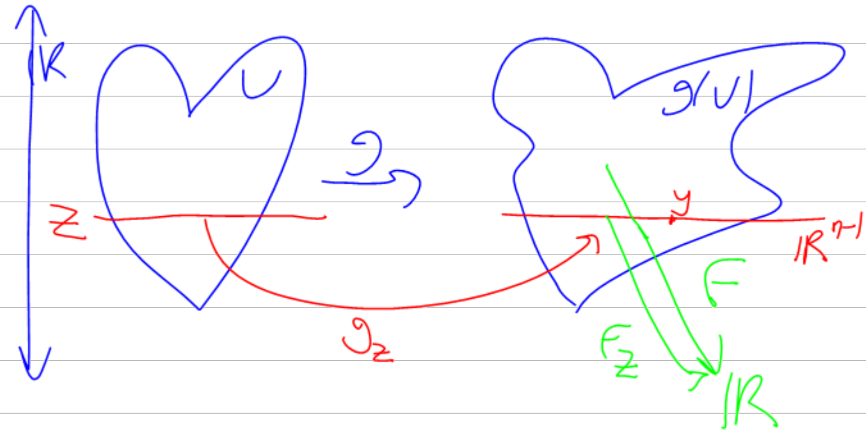
cont. diffable, 1-1, $g'(x)$ invertible & $g(U)$ is bndd. Then a restricted COV holds. Namely, if $F : g(U) \longrightarrow \mathbb{R}$ is bndd & cont. & $\operatorname{supp} F \subset g(U)$, then

$$\int_{\mathbb{R}^n \cancel{g(U)}} F = \int_{\cancel{U}\ \mathbb{R}^n} (F \circ g) |\det g'|$$

For simplicity write all integrals on $\mathbb{R}^n / \mathbb{R}^{n-1} / \mathbb{R}$ extending the integrand by 0 as necessary.

For $z \in \mathbb{R}$ define

$$g_z : \begin{pmatrix}\text{subset} \\ \text{of} \\ \mathbb{R}^{n-1}\end{pmatrix} \longrightarrow \mathbb{R}^{n-1}$$



by

$$g_z(x) = \begin{pmatrix} g_1(x,z) \\ \vdots \\ g_{n-1}(x,z) \end{pmatrix}$$

& $F_z : \begin{pmatrix}\text{subset} \\ \text{of} \\ \mathbb{R}^{n-1}\end{pmatrix} \longrightarrow \mathbb{R}$ by $F_z(y) = F(y,z)$

$$g' = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_{n-1}} & \frac{\partial g_1}{\partial z} \\ \vdots & & & \\ \frac{\partial g_{n-1}}{\partial x_1} & \cdots & \frac{\partial g_{n-1}}{\partial x_{n-1}} & \frac{\partial g_{n-1}}{\partial z} \\ \cancel{\frac{\partial g_n}{\partial x_1}}\ 0 & \cdots & \cancel{\frac{\partial g_n}{\partial x_{n-1}}}\ 0 & \cancel{\frac{\partial g_n}{\partial z}}\ 1 \end{pmatrix}$$

at $\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ z \end{pmatrix}$

For us $g_n = z$

$$= \left(\begin{array}{c|c} g_z' & \ast \\ \hline 0 & 1 \end{array}\right) \Longrightarrow \det g' = \det g_z'$$

$$\int_{\cancel{g(U)}\ \mathbb{R}^n} F = \int_{\mathbb{R}} dz \underbrace{\int_{\mathbb{R}^{n-1}} dx\, F(x,z)} = \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} F_z$$

$\underline{\underline{\text{by cov}(F_z)}}$ $\displaystyle\int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} (F_z \circ g_z) |\det(g_z')| = \int$

$$= \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} dx \, \underbrace{(F_z \circ g_z)(x)}_{\overset{(F\circ g)(x,z)}{\parallel}} \cdot |\det g'(x,z)|$$

$$= \int_{\mathbb{R}^n} (F \circ g) |\det g'| \qquad \square$$

<u>Lemma 2</u> In the conditions of the main thm, for every $a \in A$ there is some open $U \ni a$, s.t. on U g is a composition of cood. swaps & l.p. maps.

$T_{ij} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $\quad T_{ij} \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix}$ $\qquad g : \mathbb{R}^n_x \longrightarrow \mathbb{R}^n_y$

$y_i = g_i(x_1, \dots, x_n)$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \overset{\alpha_K}{\longmapsto} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_K \end{pmatrix} \overset{\beta_K}{\longmapsto} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

<u>Pf</u> Let $\alpha_K : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ g_K(x_1 \dots x_n) \end{pmatrix}$

$$\alpha_k' = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \frac{\partial g_k}{\partial x_1} & \cdots \frac{\partial g_k}{\partial x_k} & \cdots & & \frac{\partial g_k}{\partial x_n} \end{pmatrix}$$

invertible iff $\frac{\partial g_k}{\partial x_n} \neq 0$.

$$g' = \begin{pmatrix} & \frac{\partial g_1}{\partial x_n} \\ \text{---} & \vdots \\ & \frac{\partial g_n}{\partial x_n} \end{pmatrix}$$

invertible $\Rightarrow$ at least one entry in last col. is $\neq 0$.

Continue with a k for which $\frac{\partial g_k}{\partial x_n} \neq 0$.

Now α_k satisfies the cond of the IFT, so on some nbd U of a α_k is invertible.

Set $\beta_k = g \circ \alpha_k^{-1}$ on U:

$$g = \beta_k \circ \alpha_k = \tau_{k,n} \circ \underbrace{\tau_{k,n} \circ \beta_k}_{\text{l.p.}} \circ \tau_{1n} \circ \underbrace{\tau_{1n} \circ \alpha_k \circ \tau_{1n}}_{\text{l.p.}} \circ \tau_{1n}$$

$\square$