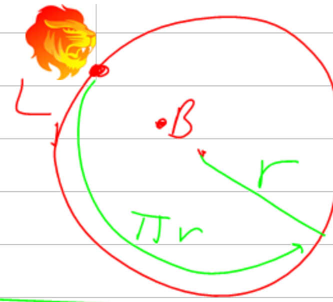
MAT257 Analysis II on November 30, 2020: Integration NT.
Read Along: Spivak 63-66 and 66-74. HW9 deadline extended to Friday!
Riddle Along: Save Ben!

$V_L = 4V_B$



POI: Given $A, \mathcal{U}$ $\exists$ φ_i loc fin, sum $= 1$, subordinate.

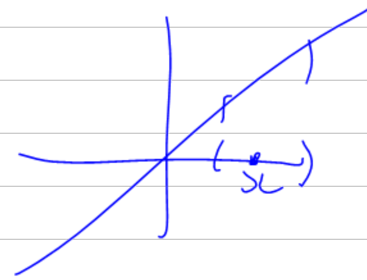
Suppose $F: A \longrightarrow \mathbb{R}$, $A \subset \mathbb{R}^n$, open, not-necessarily bndd

not nec. bndd. Yet, locally bndd. $\forall x \in A$ $\exists \epsilon > 0$ s.t. $B_\epsilon(x) \subset A$ & F is bndd on $B_\epsilon(x)$.

Also assume disc(F) is meas-0.

bndd $\Rightarrow$ loc. bndd.

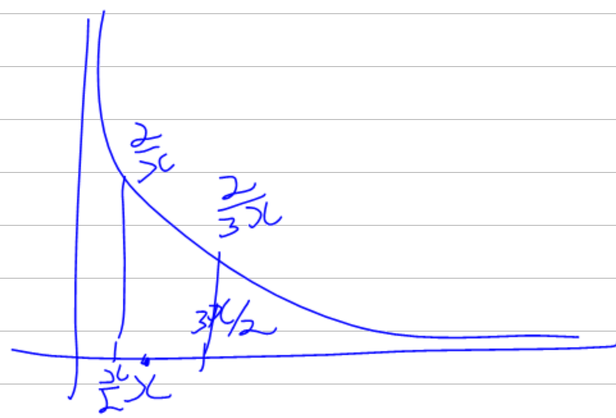
$y = x$ isn't bndd on $A = \mathbb{R}$ yet it is loc. bndd



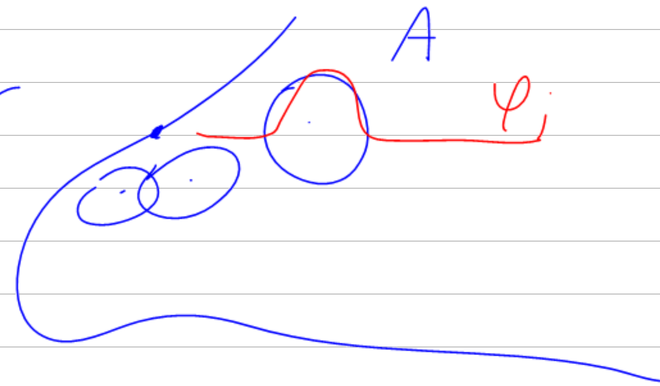
$y = \frac{1}{x}$ on $A = [x > 0]$

Given x

$B_{\frac{x}{2}}(x)$



Let $\mathcal{U}$ be some cover of A by bndd open sets on which F is bndd. Let $\Phi = \{\varphi_i\}$ be a POI for A subordinate to $\mathcal{U}$.



makes sense!

$$\int_A F = \underbrace{\left[\int_A 1 \cdot F = \int_A \left(\sum \varphi_i\right) F\right]}_{\text{in Dror's imagination}} = \sum_{i=1}^{\infty} \overbrace{\int_U \varphi_i F}^{\text{makes sense!}}$$

for some $U \in \mathcal{U}$ $\quad$ $\operatorname{supp} \varphi_i \subset U$

Alas, summation is a problem.

E.g. $\sum (-1)^n \frac{1}{n}$ converges to $\frac{\pi}{6}$

Thm (retro, I beg for forgiveness)

If a_i is a seq converging to 0,

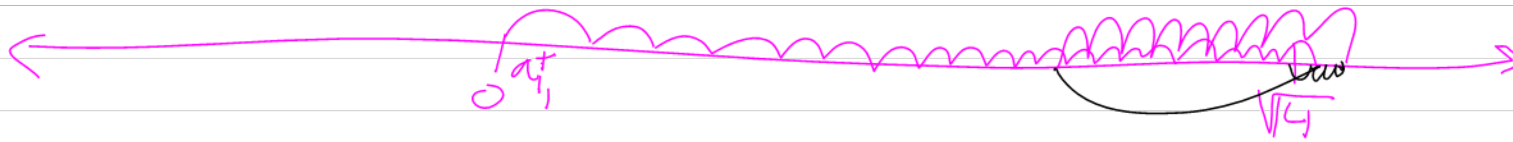
and $a_i^+ = \begin{cases} a_i & a_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$ Good

$a_i^- = \begin{cases} a_i & a_i \leq 0 \\ 0 & \text{otherwise} \end{cases}$ Evil

if $\underbrace{\sum a_i^+}_{\text{infinite good}}$ & $\underbrace{\sum a_i^-}_{\text{infinite evil}}$ both diverge,

then there is some permutation (b_i) of the (a_i)'s s.t. $\sum b_i = \sqrt{14}$.

$\xrightarrow{\text{good}}$ $\quad$ $\xleftarrow{\text{evil}}$



However, if $\sum |a_i| < \infty$ (finite evil / finite good)

$\sum a_n$ converges, & if (b_n) is a perm of (a_n)'s, $\sum b_n = \sum a_n$.

If $\sum |a_i| < \infty$ we say that (a_i) is absolutely convergent.

Def We say that f is $(\mathcal{U}, \Phi)$-integrable if $\int \varphi_i |f|$ is absolutely convergent,

$$\int \varphi_i |f| \geq \left| \int \varphi_i f \right|$$

hence $\left| \int \varphi_i f \right|$ is also absolutely conv.

In this case set

$$\int_A^{(\mathcal{U}, \Phi)} f = \sum_{i=1}^{\infty} \int_{R \text{ containing } \operatorname{supp} \varphi_i} \varphi_i f$$

Thm If $(\mathcal{U}', \Phi')$ are another cover &

another POI, satisfying same cond., & $\Phi' = \{\varphi'_j\}$. Then

$$\int_A^{(U,\Phi)} F = \int_A^{(U',\Phi')} F \qquad \left[\text{call this quantity } \int_A^{NT} F\right]$$

Proof

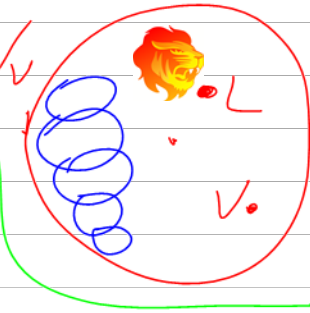
$$\int_A^{(U,\Phi)} g \overset{(1)}{=} \sum_{i=1}^{\infty} \int \varphi_i g \overset{(2)}{=} \sum_{i=1}^{\infty} \int \left(\sum_{j=1}^{\infty} \varphi'_j\right) \varphi_i g$$

$$\overset{(3)}{=} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int \varphi'_j \varphi_i g \overset{(4)}{=} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \int \varphi_i \varphi'_j g$$

$$= \sum_{j=1}^{\infty} \int \left(\sum_{i=1}^{\infty} \varphi_i\right) \varphi'_j g = \sum_{j=1}^{\infty} \int \varphi'_j g = \int_A^{(U',\Phi')} g$$

MAT257 Analysis II on December 2, 2020: Integration NT, @9:50: changes of variable.
Read Along: Spivak 63-66 and 66-74. Comment on Solution Sets!
Riddle Along: Save Vidhu!

$V_L = V_V$



$f: A \xrightarrow{\text{open}} \mathbb{R}$ not nec. bndd. locally bndd but not necessarily bndd. disc(f) of meas-0

$\mathcal{U}$: Cover A by bndd open sets contained in A. $\Phi = \{\varphi_i\}$: PoI for A subordinate to $\mathcal{U}$

f "$(\mathcal{U},\Phi)$-integrable" means $\sum \int_U^{\text{old}} \varphi_i |f| < \infty$; $\int_A^{(\mathcal{U},\Phi)} f := \sum \int \varphi_i f$

Thm $\cancel{(\mathcal{U},\Phi)}$-integrable $\Leftrightarrow$ $\cancel{(\mathcal{U}',\Phi')}$-integrable & $\int_A^{\cancel{(\mathcal{U},\Phi)}} f = \int_A^{\cancel{(\mathcal{U}',\Phi')}\,NT} f$.

NT

Pf

1.
$$\int_A^{(\mathcal{U},\Phi)} g \underset{(1)}{=} \sum_i \int_U \varphi_i g \underset{(2)}{=} \sum_i \int_U \Big(\sum_j \varphi'_j\Big)\varphi_i g \underset{(3)}{=} \sum_i \sum_j \int_{U\cap U'} \varphi'_j \varphi_i g$$

$\operatorname{supp} \varphi_i \subset U$, $\operatorname{supp} \varphi'_j \subset U'$

$$\underset{(4)}{=} \sum_j \sum_i \int_{U\cap U'} \varphi_i \varphi'_j g \underset{(3)}{=} \sum_j \int_{U'} \Big(\sum_i \varphi_i\Big)\varphi'_j g \underset{(2)}{=} \sum_j \int_{U'} \varphi'_j g \underset{(1)}{=} \int_A^{(\mathcal{U}',\Phi')} g$$

Justifications for $g = |f|$

(1) – ignore

(2) – sum = 1

$\Rightarrow$ f is $(\mathcal{U},\Phi)$-integ iff it is $(\mathcal{U}',\Phi')$-integrable.

(3) – A finite sum as supp of φ_i is compact & φ'_j is loc. finite.

(4) – because all terms are ≥ 0

Justifications for $g = f$ assuming f is NT-integrable

(1) – def.

(2) – sum = 1

(3) – A finite sum

(4) – by absolute convergence.

Thm 1. If $\overbrace{A}^{\text{also}}$ & f are bndd, ~~and supp $f \subset A$~~, then f is integ (NT).

2. If (in addition) A is Jordan-measurable, then then $\int_A^{NT} f = \int_A^{\text{old}} f$

$\underline{\text{Pf}}$ 1. Assume $|f| \leq M$, and A is contained in some rectangle R. Take some $(\mathcal{U}, \Phi)$,

$$\underbrace{\sum_{i=1}^{N} \int_R \varphi_i |f|}_{\uparrow} = \int_R \Big(\underbrace{\sum_{i=1}^{N} \varphi_i}_{\leq 1}\Big) \underbrace{|f|}_{\leq M} \leq 1 \cdot M \cdot \mathrm{Vol}(R)$$

bndd increasing seq. as a fnctn of N, so it is convergent, so $\sum_{i=1}^{\infty} \int \varphi_i |f|$ converges.

(2) It is now also given that A is Jordan measurable (bd A has meas 0) so we can find a cmpct $C \subset A$ s.t. $\mathrm{Vol}(A - C) < \epsilon$ (assuming ϵ was given in advance)

For only finitely many i's $\mathrm{supp}\,\varphi_i \cap C \neq \emptyset$

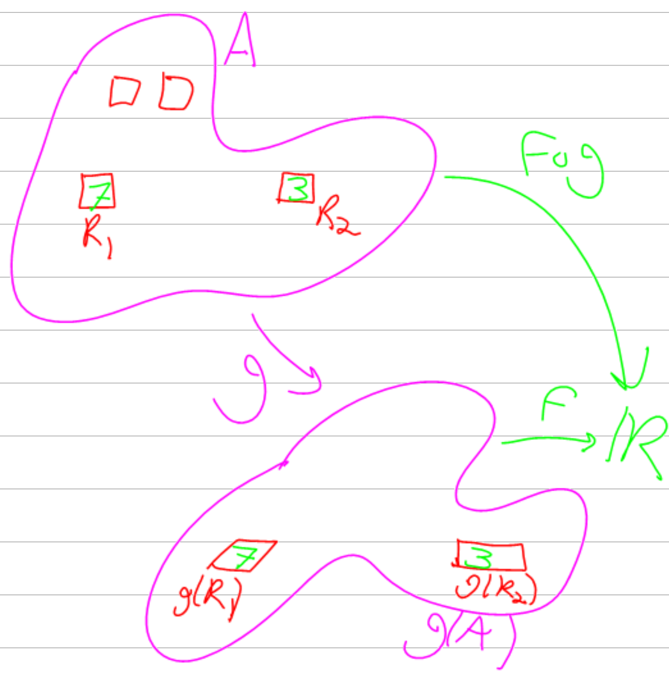
Let N be bigger than all of these i's.

$$\left| \int_A f - \sum_{i=1}^{N} \int \varphi_i f \right| = \left| \int \Big(1 - \sum_{i=1}^{N} \varphi_i\Big) f \right|$$

$$\leq \int_A \left(1-\sum_{i=1}^{N} \varphi_i\right)|F| = \int_{\underbrace{A-C}_{\text{vol} \leq \varepsilon}} \underbrace{\left(1-\sum \varphi_i\right)}_{\leq 1} \underbrace{|F|}_{\leq M}$$

$\leq \varepsilon \cdot M$ & this can be made small.

$\square$



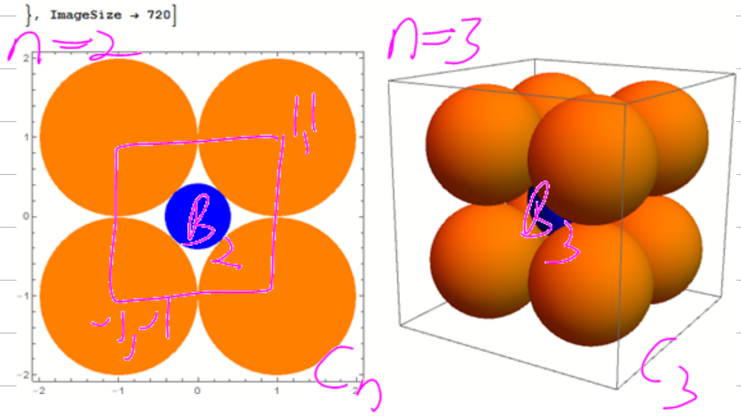
<u>Thm</u> (Change of Variables, "COV")

Let $A \subset \mathbb{R}^n$ be open, $g: A \to \mathbb{R}^n$ cont. diffable, 1-1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $F: g(A) \to \mathbb{R}$ is integrable, then

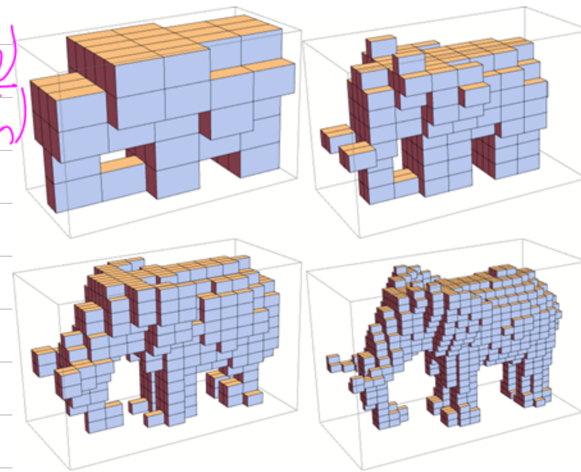
$$\int_{g(A)} F = \int_A (F \circ g) \underbrace{|\det g'|}_{\substack{\text{Jacobian} \\ \text{of } g}}$$

MAT257 Analysis II on December 4, 2020: Changes of variable, Gaussian integration, volumes of spheres and balls.
Read Along: Spivak 66-74.
Riddle Along: Compute lim_{n\to\infty}Vol(B_n)/Vol(C_n), where B_n is the largest ball bounded by 2^n balls of radius ones with centers at {-1,1}^n and C_n is the smallest cubes bounding same balls. Promise: You will learn something very surprising if you solve this riddle.

```
GraphicsRow[{
  Graphics[{Orange, Disk /@ Tuples[{1, -1}, 2], Blue, Disk[{0, 0}, √2 - 1]}, Frame -> True],
  Graphics3D[{Orange, Ball /@ Tuples[{1, -1}, 3], Blue, Ball[{0, 0, 0}, √3 - 1]}]
}, ImageSize → 720]
```

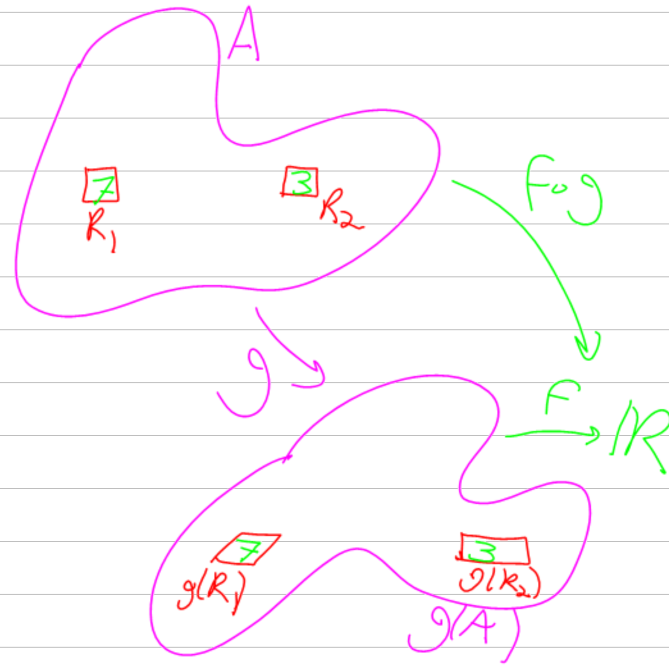


$$\lim_{n\to\infty}\frac{Vol(B_n)}{Vol(C_n)} = ?$$



Thm (Change of Variables, "COV")
Let $A\subset\mathbb{R}^n$ be open, $g: A\to\mathbb{R}^n$ cont. diff'able, 1-1, and s.t. $\forall x\in A$ $g'(x)$ is invertible. If $F: g(A)\to\mathbb{R}$ is integrable, then

$$\int_{g(A)} F = \int_A (F\circ g)|\underbrace{\det g'}_{\text{Jacobian of } g \text{ at } x}|$$



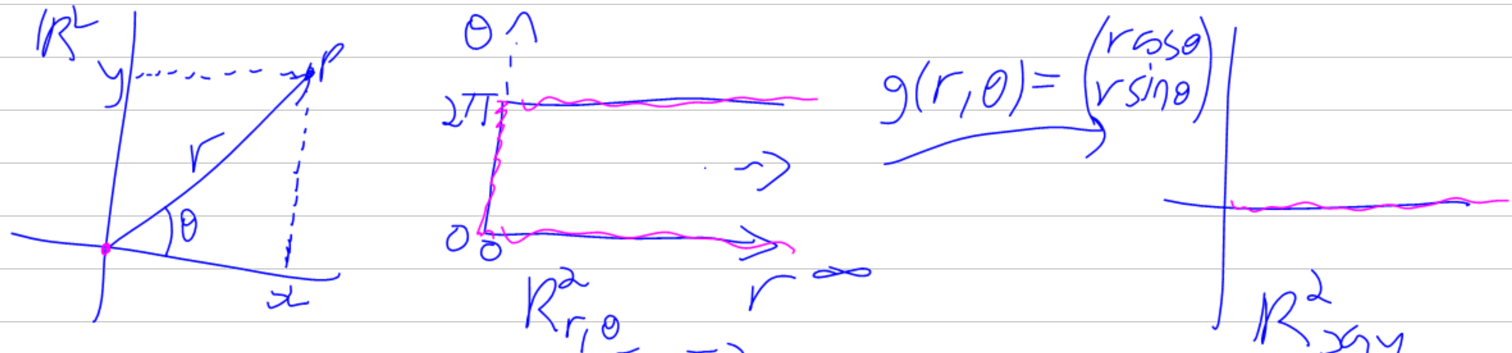
Compute $I_1 = \int_{\mathbb{R}} e^{-x^2/2}\,dx$ "most important integral in mathematics".

$$I_2 = \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\,dx\,dy \overset{(1)}{=} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy\, e^{-x^2/2} e^{-y^2/2}$$

$$\boxed{(2)\quad \int^{NT} F7 = \sum \int \varphi_i F\,7 \overset{7\int^{NF}}{=} 7\sum \int \varphi_i F}$$

$$\overset{(2)}{=} \int_{\mathbb{R}} dx\, e^{-x^2/2} \int_{\mathbb{R}} dy\, e^{-y^2/2} = I_1 \int_{\mathbb{R}} dx\, e^{-x^2/2} = I_1^2$$

Compute I_2 by switching to "polar coordinates"



$A=[0,\infty)\times[0,2\pi)$

$$\operatorname{Jac}(g)=\det g'=\det\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}=r$$

$$I_2\overset{(3)}{=}\int_A dr\,d\theta\, e^{-r^2/2}\cdot r$$

$$\int_{g(A)} \overset{e^{-(x^2+y^2)/2}}{\overset{\shortparallel}{F}} = \int_A (F\circ g)\,\underbrace{|\det g'|}_{\text{Jacobian of } g \text{ at } x}$$

$$=\int_0^{2\pi} d\theta \int_0^{\cancel{\infty}\ R} e^{-r^2/2}\, r\,dr \quad \left(\lim_{R\to\infty}\right)$$

$$(F\circ g)(r,\theta)=e^{-[(r\cos\theta)^2+(r\sin\theta)^2]/2}$$

$$=e^{-r^2/2}$$

$$\overset{(4)}{=}\int_0^{2\pi} d\theta\left[-e^{-r^2/2}\right]_0^{\infty}$$

$$\left(e^{-r^2/2}\right)'=e^{-r^2/2}(-r)$$

$$=\int_0^{2\pi} d\theta\cdot 1=2\pi \implies I_1=\sqrt{2\pi}$$

~~Fubini for NT: Suppose $A\subset\mathbb{R}^n$ & $B\subset\mathbb{R}^m$ are open suppose $F: A\times B\to\mathbb{R}$ loc. bdd, disc(F) of meas-0.~~

~~$$\int_{A\times B} F \sim \int_A dx \int_B dy\, F$$~~

added Dec 2, 2021: Wrong! There is no simple-minded Fubini for $\int^{NT}$!

Suppose $(\mathcal{U}, \Phi)$ are a cover, PoU for A

Suppose $(\mathcal{V}, \Psi)$ are a cover, PoU for B

Then $\mathcal{W} = \left\{ U \times V : \begin{array}{l} U \in \mathcal{U} \\ V \in \mathcal{V} \end{array} \right\}$ is an approp. cover of $A \times B$

& $\Phi \times \Psi = \{ \varphi_i(x)\psi_j(y) \}$ is a PoU for $A \times B$ sub. to $\mathcal{W}$

$$\int_{A\times B}^{NT} f = \sum_{i,j} \int^{old} \varphi_i(x)\psi_j(y) f(x,y)\,dx\,dy \qquad \left[\text{assume } f \text{ is cont.}\right]$$

$$= \sum_i \int^{old} dx\, \varphi_i(x) \int^{old} dy \sum_j \psi_j(y) f(x,y)$$

$$= \sum_i \int^{old} dx\, \varphi_i(x) \int_B^{NT} dy\, f(x,y) = \int_A^{NT} dx \int_B^{NT} dy\, f$$