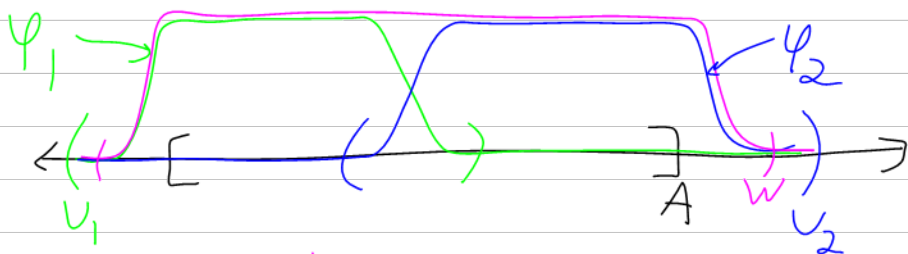
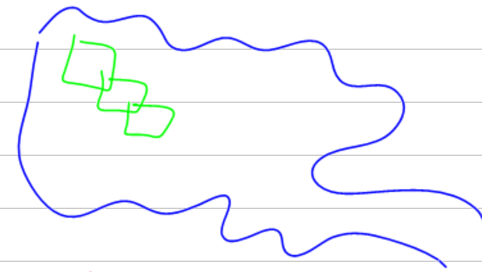
$\vec{1}\ \vec{2}\ \vec{3}\ \vec{4}\ \vec{5}\ \vec{6}$

MAT257 Analysis II on November 23, 2020: Partitions of unity.
Read Along: Spivak 63-66. Note the change in schedule!
Riddle Along: n b/w-hat-wearing prisoners stand in a row; each one sees the hats ahead of them but not their own or the ones behind. They each must guess and shout the colour on their head, going from the back forward. If more than one is wrong, all are executed. Could they have devised a strategy in advance, to save themselves?





A way to divide labour: $1_A \le \varphi_1+\varphi_2 \le 1_W$

φ_i smooth, $\operatorname{supp}\varphi_i \subset U_i$

Later: $f: A \to \mathbb{R}$

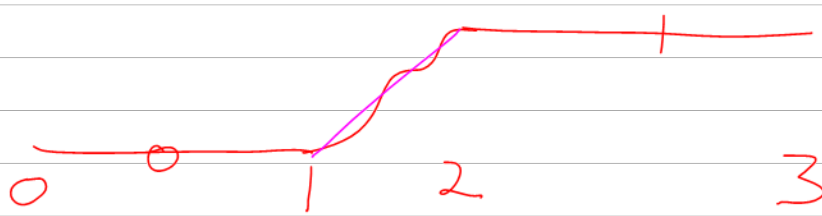
$$\int_A^{NT} f := \int_{U_1} \varphi_1 f + \int_{U_2} \varphi_2 f$$

Thm (PO1) Given $A \subset \mathbb{R}^n$ & $\mathcal{U}$ an open cover thereof, we can find a countable collection $\Phi = \{\varphi_i : W \to [0,1]\}$ of C^∞ functions defined on some open $W \supset A$, s.t.

1. Φ is locally finite: Each $x \in W$ has some open neighborhood $V \ni x$, s.t. "loc. fin"
$$|\{i : \operatorname{supp}(\varphi_i) \cap V \neq \emptyset\}| < \infty$$

2. $\forall x \in A \quad \sum \varphi_i(x) = 1$. "Sum=1"

3. $\forall \varphi_i \in \Phi \ \exists U \in \mathcal{U}$ s.t. $\operatorname{supp}\varphi_i \subset U$. "subordinate"



Mt. Connor, Aus

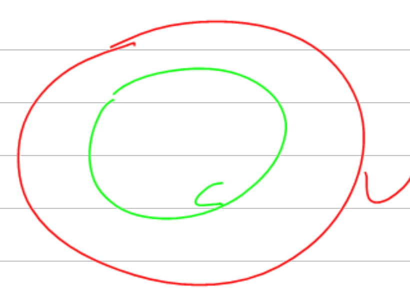
$e^x \quad \sin x$ is C^∞

$$|x| = \begin{cases} x & x \ge 0 \\ -x & x \le 0 \end{cases}$$

Prel 1 $\exists$ smooth flat-top mountains,

If $\underset{\text{compact}}{C} \subset \underset{\text{open}}{U} \subset \mathbb{R}^n$

then $\exists f \in C^\infty(\mathbb{R}^n)$ s.t.

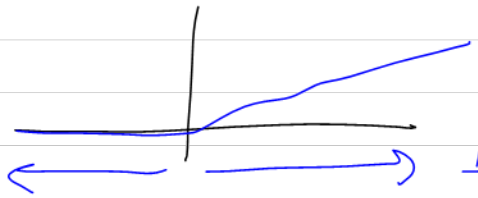


$F|_C \equiv 1 \quad \operatorname{Supp} F \subset U.$

Step 1 $\exists$ smooth 1D "seashores"

$\sigma \in C^\infty(\mathbb{R}) \quad \sigma(x) = 0 \text{ if } x \leq 0$

$\sigma(x) > 0 \text{ if } x > 0$



Pf

$$\sigma(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x} & x > 0 \end{cases}$$

At $x > 0$

$$\sigma'(x) = \frac{1}{x^2} \cdot e^{-1/x} \xrightarrow[x \to 0]{} 0$$

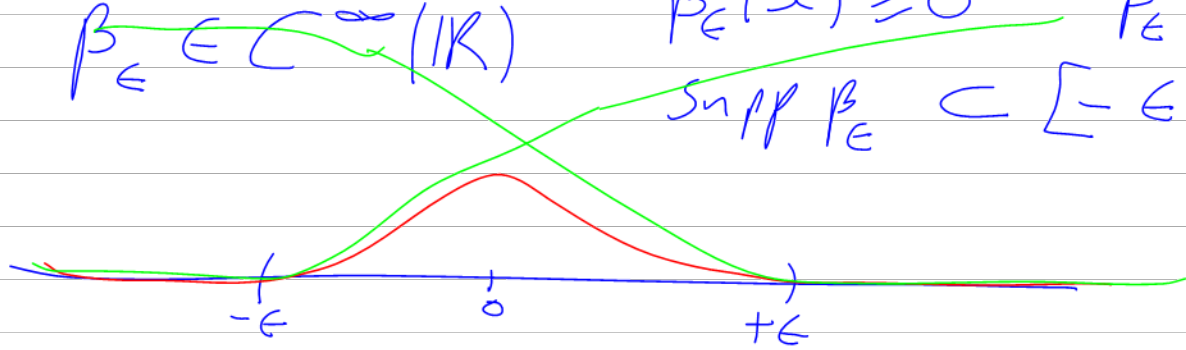
$$\sigma''(x) = \left(-\frac{1}{2x^3} + \frac{1}{x^2} \cdot \frac{1}{x^2} \right) e^{-1/x} \xrightarrow[x \to 0]{} 0$$

$$\sigma^{(n)}(x) = \left(\begin{matrix} \text{rational} \\ \text{fnctn} \end{matrix} \right) \cdot e^{-1/x} \xrightarrow[x \to 0]{} 0$$

2. smooth 1D bumps.

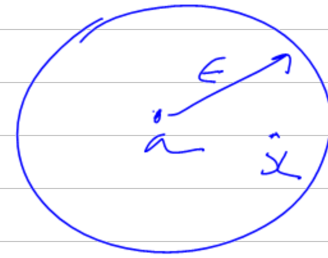
$\beta_\epsilon \in C^\infty(\mathbb{R}) \quad \beta_\epsilon(x) \geq 0 \quad \beta_\epsilon(0) > 0$

$\operatorname{Supp} \beta_\epsilon \subset [-\epsilon, \epsilon]$



Pf $\quad \beta_\epsilon(x) = \sigma(x+\epsilon)\,\sigma(\epsilon - x)$

3. $\exists$ nD bumps $\beta_{a,\epsilon} = \beta$



$$\beta(a) > 0 \qquad \beta(x) \geq 0$$

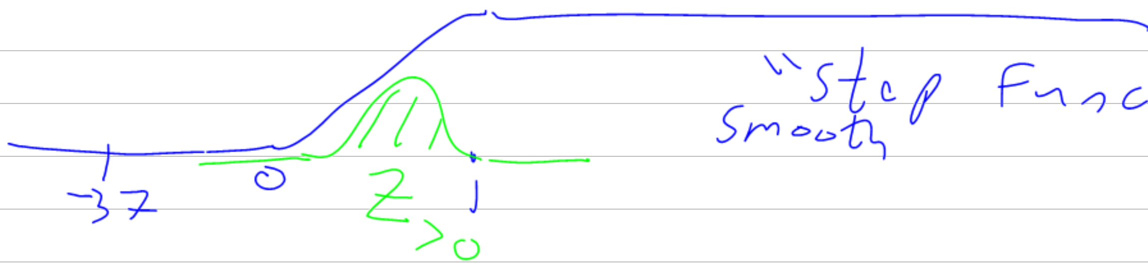
$$\operatorname{supp} \beta \subset B_\epsilon(a)$$

guess

$$\beta_{a,\epsilon}(x) = \beta_\epsilon(|x-a|)$$

$$\beta_{a,\epsilon}(x) = \beta_{\epsilon^2}\left(\sum (x_i - a_i)^2\right) \qquad |x-a|^2 = \sum (x_i - a_i)^2$$

4.



smooth "step function"

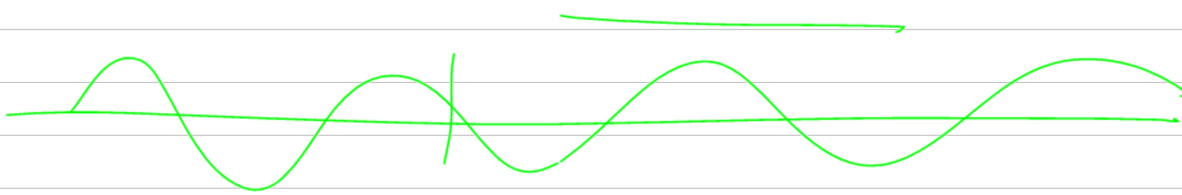
$$\theta \in C^\infty(\mathbb{R}) \qquad \theta(x) = 0 \quad x \leq 0$$

$$\theta(x) = 1 \quad x \geq 1.$$

$$z = \int_0^1 \beta_{1/2,1/2}(t)\,dt$$

Pf

$$\theta(x) = \frac{1}{z}\int_{-37}^{x} \beta_{1/2,1/2}(t)\,dt$$

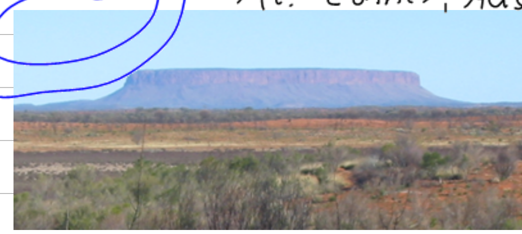


MAT257 Analysis II on November 25, 2020: Partitions of unity.
Read Along: Spivak 63-66. Note the newer change in schedule! HW8 due tonight; HW9 posted.
Riddle Along: Me? Riddles? ← make a list!

POI: Given $A, \mathcal{U}$ $\exists \varphi_i$ loc fin, sum $=1$, subordinate.

<u>Prop. 1</u> $\exists$ smooth flat-top mountains:

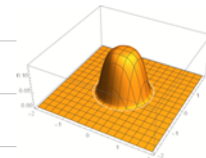
Mt. Conner, Aus



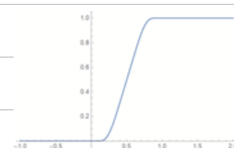
If $C \subset U \subset \mathbb{R}^n$, $\exists f \in C^\infty(\mathbb{R}^n)$ s.t.
compact open

<u>Steps</u> $f|_C \equiv 1$, $\operatorname{supp} f \subset U$

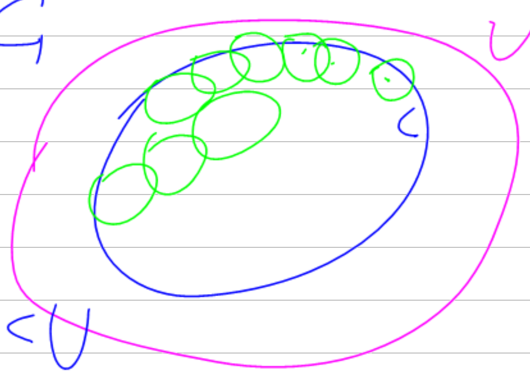
3. $\exists$ smooth nD bumps $\beta_{a\epsilon}(a) > 0$, $\beta_{a\epsilon}(x) = 0$ $|x-a| > \epsilon$



4. $\exists$ smooth 1D steps $\theta(x) = 0$ $x \leq 0$, $\theta(x) = 1$ $x \geq 1$



Step 5: last. For each $x \in C$



Find ϵ s.t. $B_\epsilon(x) \subset U$

Find a fctn $\beta_x \geq 0$,

$\beta_x(x) > 0$, $\operatorname{supp} \beta_x \subset B_\epsilon(x) \subset U$

Let $U_x = \{y : \beta_x(y) > 0\} = \beta_x^{-1}((0, \infty))$

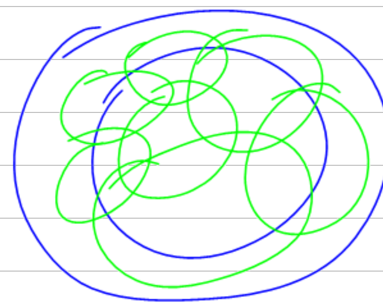
an open set,

$x \in U_x$

So $\{U_x\}$ covers C, by compactness

Find $x_1 \ldots x_n$ s.t. $\bigcup_{i=1}^n U_{x_i} \supset C$. Set

$$\varphi(x) = \sum_{i=1}^n \beta_{x_i}(x)$$



φ is smooth. $\varphi \geq 0$, $[\varphi > 0] = \{x : \varphi(x) > 0\}$

$$= \bigcup_{i=1}^{n} [\beta_{x_i} > 0] = \bigcup U_{x_i} \supset C_1$$

<u>Aside</u> A cont. fnctn on a cmpct set attains its min & max; namely if $\varphi : \underset{\text{cmpct}}{C_1} \to \mathbb{R}$ is cont., $\exists\, a, b \in C$ s.t.

$$f(a) \leq f(x) \leq f(b) \quad \forall x \in C_1$$

<u>Pf</u> $f(C)$ is cmpct, so closed, so

$$\underset{\substack{\| \; \exists a \in C \\ m = f(a)}}{\inf f(C)} \in f(C_1) \qquad \underset{\substack{\| \; \exists b \in C_1 \\ M = f(b)}}{\sup f(C)} \in f(C)$$

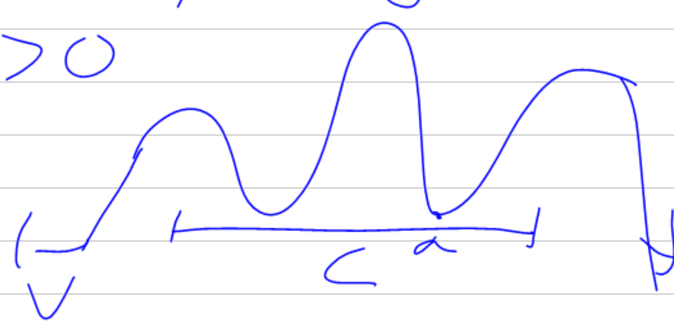
$$f(a) = m \leq f(x) \leq M = f(b)$$

---

our φ attains its min, so $\exists\, a \in C_1$ s.t. $\varphi(x) \geq \varphi(a) > 0$

Set

$$f(x) = \theta\left(\frac{\varphi(x)}{\varphi(a)}\right)$$



on C, $\varphi(x) \geq \varphi(a)$, so $\theta\left(\frac{\varphi(x)}{\varphi(a)}\right) = 1$.

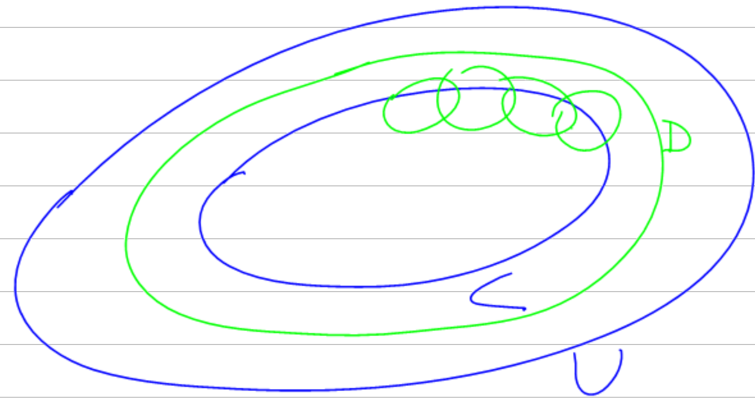
away from U, $\varphi(x)=0$ so $\theta\left(\frac{\varphi(x)}{\varphi(h)}\right)=\theta(0) = 0$

as a composition of C^∞ fnctns, F is C^∞ $\square$

<u>Prel 2</u> Given

$C \subset U \subset \mathbb{R}^n$
cmpct, open

there exists a cmpct D s.t.



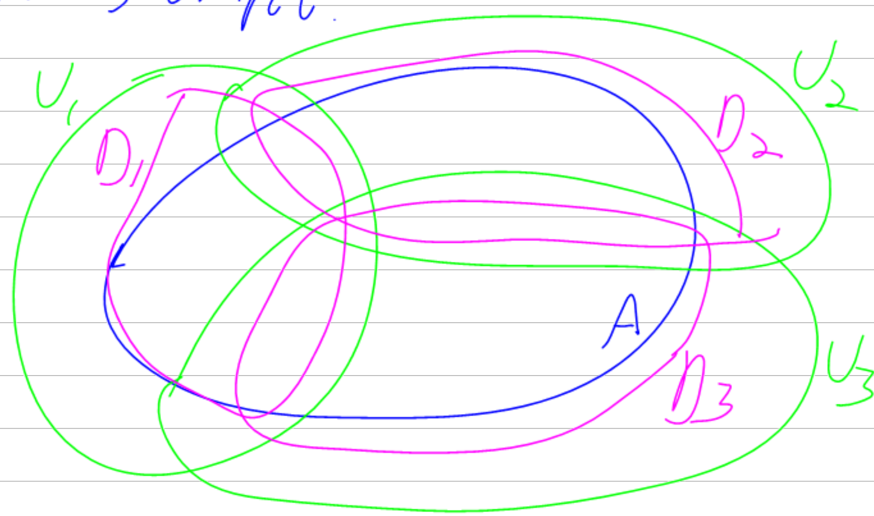
$C \subset \operatorname{int} D \subset D \subset U$. <u>Pf</u> See HW 1 or HW 2.

<u>Thm PO1</u> $A, \mathcal{U}, \exists(\varphi_i)$

1. loc. finite
2. sum $=1$ on A
3. Subordinate

<u>Pf</u> <u>step 1</u> If A is cmpct.

As A is cmpct, find $U_1 \dots U_n \in \mathcal{U}$ s.t. $\bigcup U_i \supset A$.



Consider

$$A \setminus \bigcup_{i=2}^{n} U_i \subset U_1$$

by prel 2, find $A \setminus \bigcup_2^n U_i \subset \operatorname{int} D_1 \subset D_1 \subset U_1$

to find D_k, assume $D_1 \dots D_{k-1}$ where already constructed and $\bigcup_{i=1}^{k-1} \operatorname{int} D_i \cup \bigcup_{i=k}^{n} U_i$

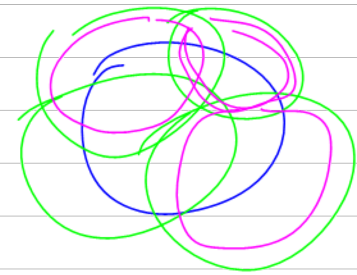
and D_i is cmpct & $D_i \subset U_i$, $\supset A$,

$$A \setminus \left(\bigcup_{i=1}^{k-1} \operatorname{int} D_i \cup \bigcup_{i=k+1}^{n} U_i \right) \subset \underset{\uparrow \text{find}}{D_k} \subset U_k$$

we found D_i cmpct, $D_i \subset U_i$,

$$\bigcup \operatorname{int} D_i \supset A$$

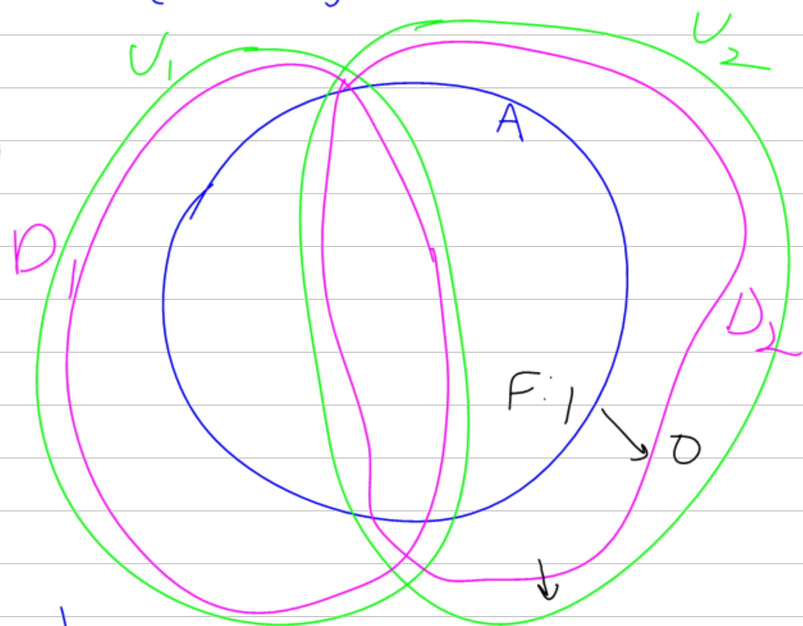
MAT257 Analysis II on November 27, 2020: Partitions of unity, integration NT.
Read Along: Spivak 63-66.
Riddle Along: Can you find irrational x and y such that x^y is rational?



POI: Given $A, \mathcal{U}$ $\exists \varphi_i$ loc fin, sum $=1$, subordinate.

Case I A is cmpct.

Pf WLOG $\mathcal{U} = \{U_i\}_1^n$ is finite. Shrink U_i to a cmpct $D_i \subset U_i$ s.t. $\{\operatorname{int} D_i\}$ covers A.



By prev. work, Find $\psi_i: \mathbb{R}^n \to [0,1]$ smooth, $\psi_i|_{D_i} \equiv 1$, $\operatorname{supp} \psi_i \subset U_i$

Define

$$\varphi_i(x) = \begin{cases} F(x) \dfrac{\psi_i(x)}{\sum \psi_i(x)} \\ 0 & x \notin \bigcup \operatorname{int} D_i \end{cases}$$

where F is a "flat t-p" $F|_A \equiv 1$, $\operatorname{supp} F \subset \bigcup \operatorname{int} D_i$

☺

Case II $A = \bigcup_{k=0}^{\infty} A_k$, $A_0 = \emptyset$, A_k is cmpct, $A_k \subset \operatorname{int} A_{k+1}$.

Let $\mathcal{U}_K =$

$\{(U \cap \operatorname{int} A_{K+2}) - A_{K-1} : U \in \mathcal{U}\}$

is a cover of

$$B_K = A_{K+1} - \operatorname{int} A_K$$

a cmpct set.

Let Φ_K be a POI for B_K subordinate to $\mathcal{U}_K$. Take $\Phi = \bigcup \Phi_K$ (still countable)

$$= \{\psi_i\}$$

Set

$$\varphi_i(x) = \frac{\psi_i(x)}{\sum \psi_i(x)}$$

finite sum $\neq 0$

here loc finiteness ✓

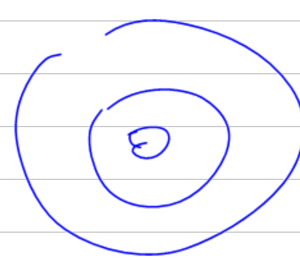
sum = 1 ✓

subordinate ✓

smooth on A. (as we never divide by 0)

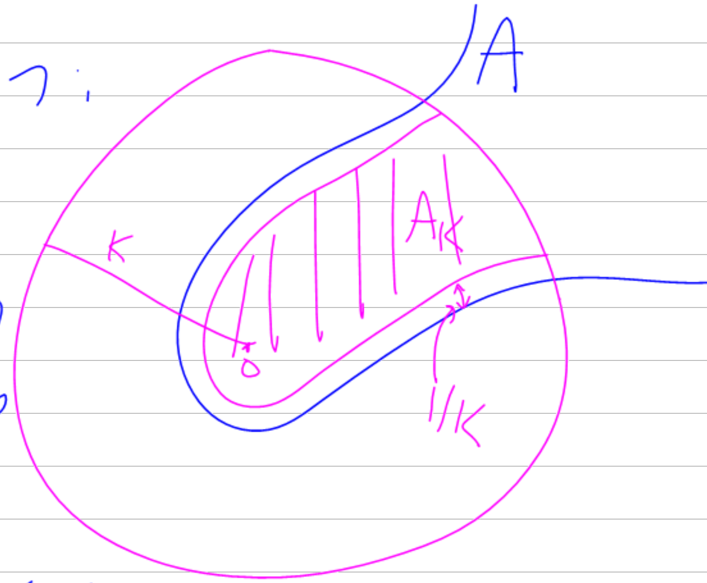
$$A = \bigcup A_K = \bigcup \operatorname{int} A_{K+1}$$

is open ✓



$A_K \subset \operatorname{int} A_{K+1}$

Case III A is open:



$A = \bigcup A_k$

$A_k = \left\{ x \in A : \begin{array}{l} |x| \leq k, \\ \mathrm{dist}(x, A^c) \geq \frac{1}{k} \end{array} \right\}$

Case IV A is arbitrary.

Replace A by $\bigcup_{U \in \mathcal{U}} U = B$ $\left(\begin{array}{c}\text{the union of} \\ \text{all sets in} \\ \mathcal{U}\end{array}\right)$

by Case III, find a POI $\Phi = \{\varphi_i\}$

for B subordinate to $\mathcal{U}$,

this POI is also good for A.

$\boxed{☺}$