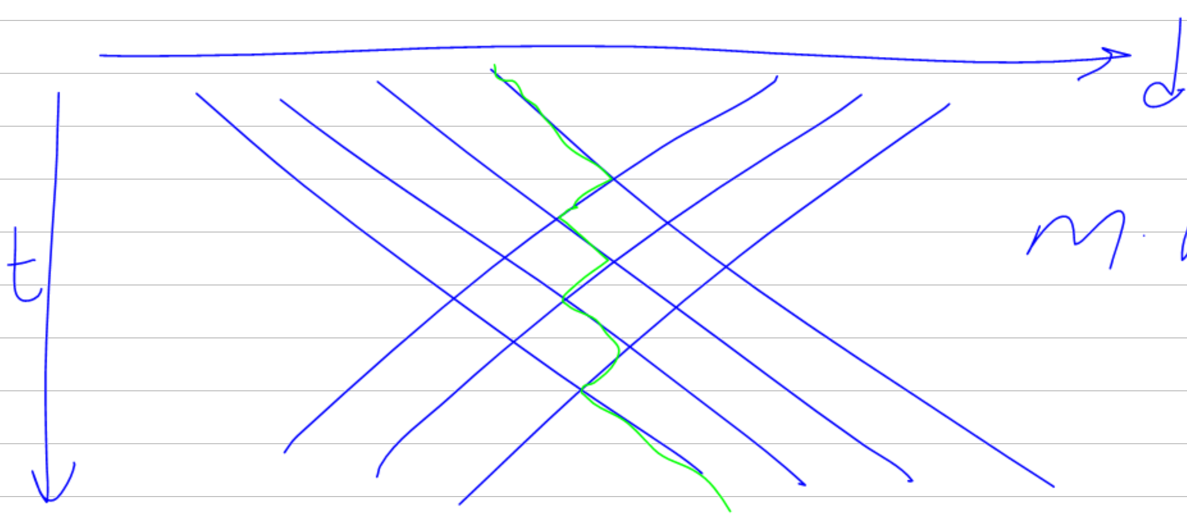
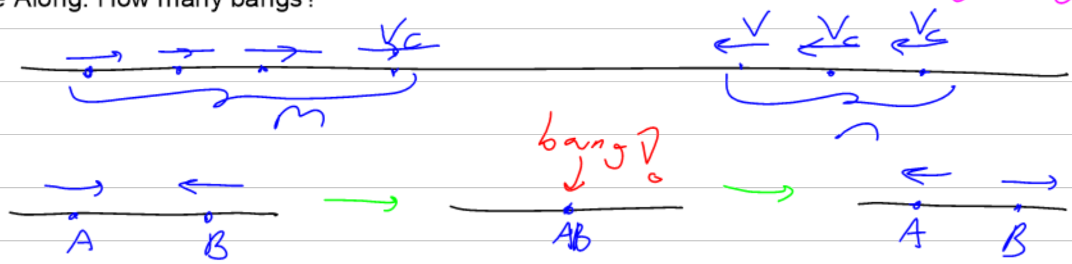


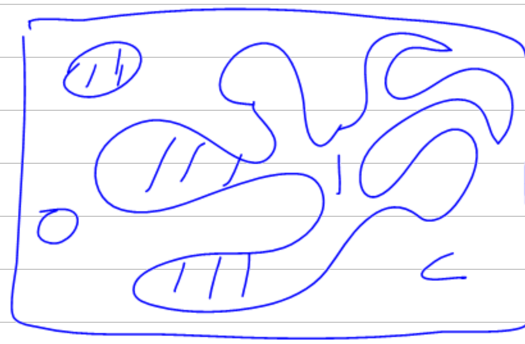

m·n

MAT257 Analysis II on November 16, 2020: Fubini's Theorem.
Read Along: Spivak 55-56, 56-62.
TT1 Comments: Solution sets, appeals.
Riddle Along: How many bangs?

Q2: Dror
else: Shaoyang.




If $C \subset R$ is any subset of a rectangle $R$:




$$\chi_C(x) = 1_C(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C \end{cases}$$

the characteristic function of $C$

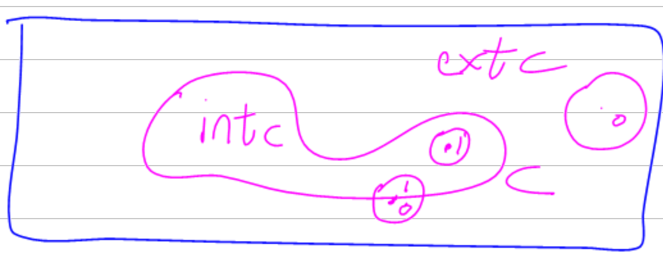
$$\mathrm{Vol}(C) = \int_R 1_C$$

if exists?!

"content" "volume"
2D: "area"
1D: "length"

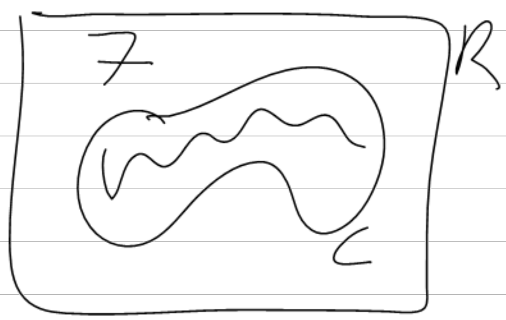
<u>Claim</u> $\mathrm{Vol}(C)$ makes sense ($1_C$ integrable) $\iff$ $\mathrm{bd}\,C$ is of measure 0.

<u>Pf</u> $1_C(x)$ is cont. at $x$, iff $x \notin \mathrm{bd}\,C$. So $\mathrm{disc}(1_C) = \mathrm{bd}\,C$. $\square$



$\int_R F$

**Def** If $F : C \to \mathbb{R}$ is a bndd fnctn on a bndd set,




$$\int_C F = \int_R \chi_C \cdot \tilde{F}$$

$\tilde{F}$: F extended to R.

if rhs exists.

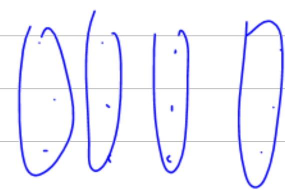
$\int F$ makes sense whenever disc(F) $\cap$ C is of meas 0.


Note that even if C is open, $\int_C (\text{cont.})$ may not make sense.

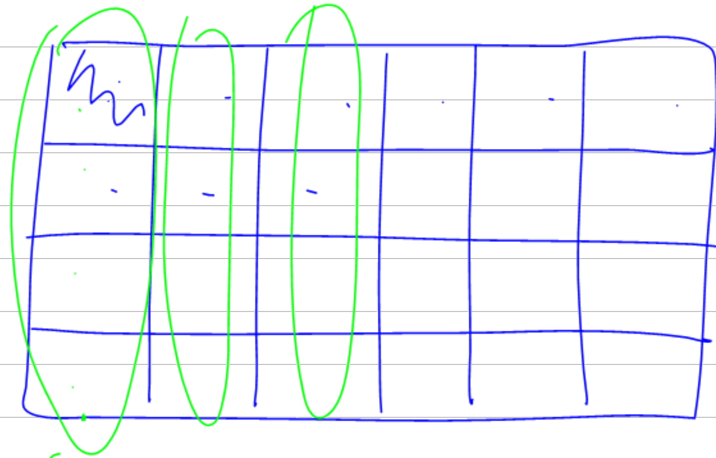
---

$$\sum_{(x,y)\in A\times B} F(x,y) = \sum_{x\in A}\left(\sum_{y\in B} F(x,y)\right)$$

where A & B are finite



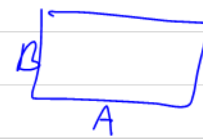
<u>Def</u> $f: R \to \mathbb{R}$
(bnd)



$$\int_R f = \underline{\int_R} f = \sup_P L(f,P)$$

$$\int_R f = \overline{\int_R} f = \inf_P U(f,P) \quad w \cdot \int f\,dy$$

$$\sum w \cdot g \to \int g \qquad \underline{\underline{g(x) = \int f(x,y)\,dy}}$$




<u>Thm</u> (not really Fubini's)

Suppose $A \subset \mathbb{R}^n_x$, $B \subset \mathbb{R}^m_y$ are rectangles.

Set $R = A \times B$ a rectangle in $\mathbb{R}^{n+m}$

Suppose $f: R \to \mathbb{R}$ bnd. $(f(x,y))$

Set

$$\overline{g}(x) = \overline{\int_B} f(x,y)\,dy \qquad \underline{g}(x) = \underline{\int_B} f(x,y)\,dy$$

If $f$ is integrable, then $\underline{g}, \overline{g}$ are also integrable &

$$\int_R f\,dx\,dy = \int_A \underline{g}\,dx = \int_A \overline{g}\,dx$$

$$\qquad\qquad \| \qquad\qquad\qquad \|$$

$$\int_A dx \left( \underline{\int_B} dy\, f(x,y) \right) \qquad \int_A dx \left( \overline{\int_B} dy\, f(x,y) \right)$$

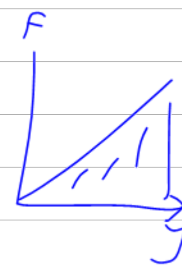
Comments/examples:

1. If $f$ is cont, $\underline{g}(x) = \overline{g}(x)$ and so

$$\int_{A \times B} f\, dx\, dy = \int_A dx \int_B dy\, f(x,y)$$

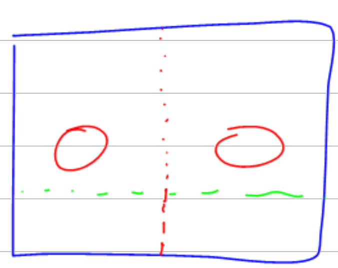
2. $n = m = 1$ $\quad f(x,y) = xy \quad R = [0,1]^2$

$$\int_{[0,1]^2} xy\, dx\, dy = \int_{[0,1]} dx \left( \int_{[0,1]} dy\, x \cdot y \right)$$




$$= \int_{[0,1]} dx \left( x \cdot \frac{1}{2} \right) = \left[ \frac{1}{4} x^2 \right]_0^1$$

$$= \frac{1}{4}.$$

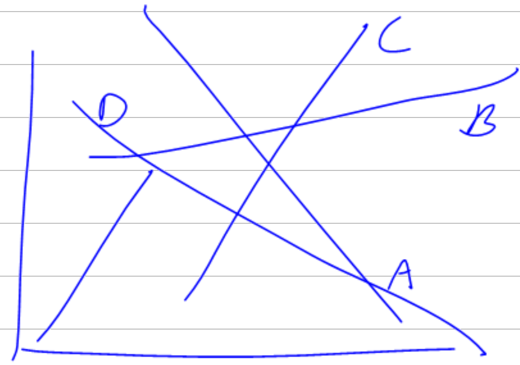
3. 


$$f(x,y) = \begin{cases} 1 & x = \frac{1}{2},\ y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

F is integrable

yet $\int F(x,y)\,dy$ exists if $x \neq \frac{1}{2}$
but not if $x = \frac{1}{2}$

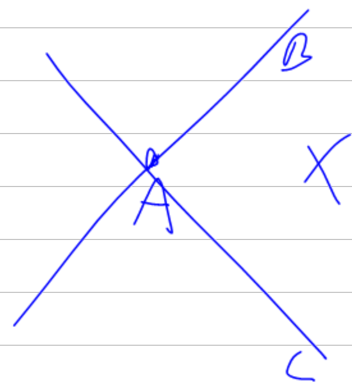
A B C D

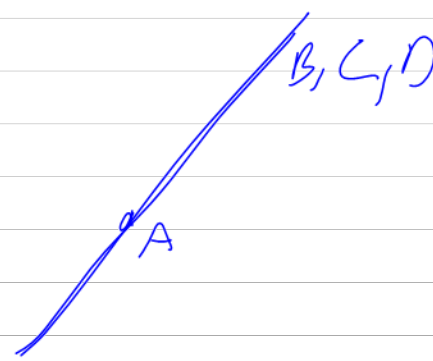



A meets B $\quad A, B \subset P_1$

$A, C \subset P_2$

B meets C $\Rightarrow P_1 = P_2$



MAT257 Analysis II on November 18, 2020: Fubini's Theorem.
Read Along: Spivak 56-62. HW7 is due at 11:59PM, HW8 is on web.
Riddle Along: Your turn!

Guido!

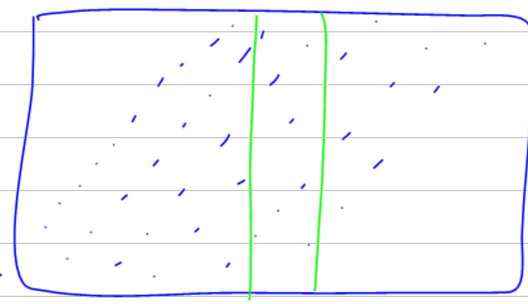
**Thm** $f:(R=A\times B\subset\mathbb{R}^n\times\mathbb{R}^m)\longrightarrow\mathbb{R}$ integrable,

$$\underline{g}(x):=\underline{\int_B} f(x,y)\,dy \qquad \bar{g}(x):=\overline{\int_B} f(x,y)\,dy$$

then $$\int_R f\,dx\,dy=\int_A \underline{g}(x)\,dx=\int_A \bar{g}(x)\,dx$$

**Comments** 1. makes integrals computable!

2. Converse does not hold



(Thanks!) $\epsilon\notin\mathbb{Q}$

$$A=\begin{pmatrix}1&\epsilon\\0&1\end{pmatrix}(\mathbb{Q}\times\mathbb{Q})$$

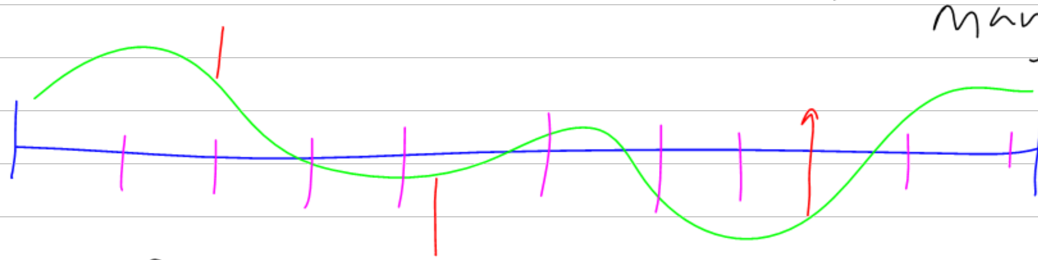
$$\chi_A$$

3. $\bar{g},\ \underline{g}$

$$g(x):=\begin{cases}\int_B dy\, f(x,y) & \text{if exists}\\ \cancel{x},\ 0 & \text{otherwise}\end{cases}$$

a. If $\bar{g}(x) = \underline{g}(x)$ except for finitely many points, then

$g(x) = \bar{g}(x) = \underline{g}(x)$ except finitely many pts.,



$$\int_A g = \int_A \underline{g} = \int_A \bar{g}$$

Naive Fubini holds:

$$\int_{A\times B} f = \int_A dx \left( \int_B dy\, f(x,y) \right)$$

b. on $[0,1]^2$

$$f(x,y) = 1 + \begin{cases} 1/q & \text{if } x,y \in \mathbb{Q},\ x = p/q \\ 0 & \text{otherwise} \end{cases}$$

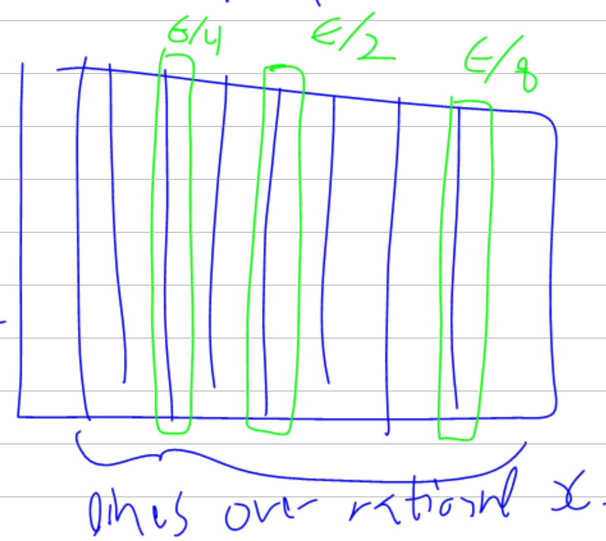
$$\bar{g}(x) = \begin{cases} 1 + \frac{1}{q} & x = p/q \\ 1 & x \notin \mathbb{Q} \end{cases}$$

$$\underline{g}(x) = \{ 1$$

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

F is integrable!

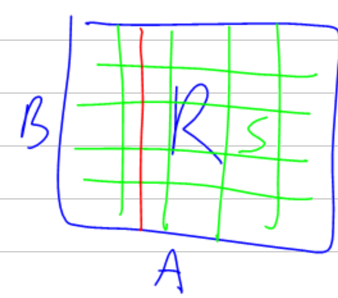
disc(F) =




$$\int_{[0,1]^2} F = \int_{[0,1]} \underline{g} = \int_{[0,1]} \overline{g} = 1$$

$$\int_{[0,1]} g \quad \text{does not exist!}$$

<u>Proof</u> (of Fubini): Let P be a partition of R

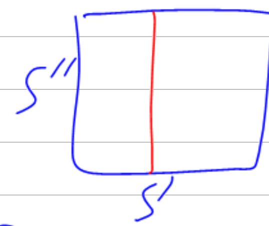


$P = P_A \times P_B$ $\qquad S \in P \qquad S = S' \times S''$

$\qquad\qquad\qquad\qquad\qquad\qquad S' \in P_A \quad S'' \in P_B$

$$L(F,P) = \sum_{S\in P} v(S)\cdot \inf_S F$$

$$= \sum_{S'\in P_A}\sum_{S''\in P_B} \underset{\substack{\| \\ v(S')\cdot v(S'')}}{v(S'\times S'')} \cdot \inf_{S'\times S''} F$$




$$= \sum_{S'\in P_A} v(S')\cdot \sum_{S''\in P_B} v(S'')\cdot \inf_{S'\times S''} F$$

$$\le \sum_{S'\in P_A} v(S') \sum_{S''\in P_B} \underbrace{\int_{S''} F(x,y)\,dy}_{\text{for any } x\in S'}$$

So

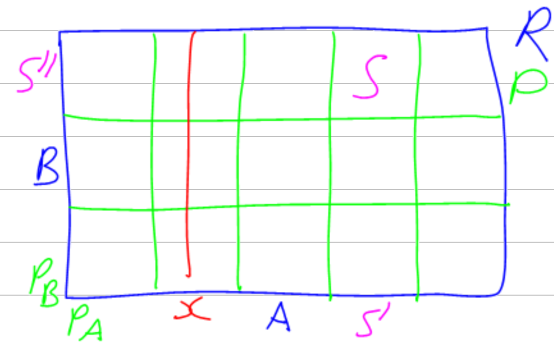
$$L(F,P) \le \sum_{S'\in P_A} v(S') \sum_{S''\in P_B} \inf_{S''}$$

MAT257 Analysis II on November 20, 2020: Fubini's Theorem, partitions of unity.
Read Along: Spivak 56-62 and 63-66.
Riddle Along: n prisoners. Each wears a tower of infinitely-many randomly-chosen b/w hats. Simultaneously each needs to point at a black hat on their head. How can they maximize the chance that they will all get it right? Can they do better than 1/2^n?.

**Thm** $f:(R=A\times B\subset\mathbb{R}^n\times\mathbb{R}^m)\to\mathbb{R}$ integrable, $\underline{g}(x):=\underline{\int_B} f(x,y)\,dy$

$$\Rightarrow \int_R f\,dx\,dy=\int_A \underline{g}(x)\,dx=\int_A \overline{g}(x)\,dx \qquad \overline{g}(x):=\overline{\int_B} f(x,y)\,dy$$

**Proof** Given $P=P_A\times P_B$, write each $S\in P$ as $S=S'\times S''$, $S'\in P_A$, $S''\in P_B$.




$$L(f,P)=\sum_{S'\in P_A} v(S')\sum_{S''\in P_B} v(S'')\inf_{S'\times S''} f$$

$$=\sum_{S'\in P_A} v(S')\sum_{S''\in P_B}\inf_{x\in S'} v(S'')\inf_{y\in S''} f(x,y)$$

$$\le\sum_{S'\in P_A} v(S')\inf_{x\in S'}\underbrace{\sum_{S''\in P_B} v(S'')\inf_{y\in S''} f(x,y)}_{L(f(x,-),P_B)\ \le\ \underline{g}(x)}$$

Aside $\inf_x h_k(x)\le h_k(x)$

$\sum_k \inf_x h_k(x)\le\sum_k h_k(x)$

$\sum_k \inf_x h_k(x)\le\inf_x\sum_k h_k(x)$

$$\le\sum_{S'\in P_A} v(S')\inf_{x\in S'}\underline{g}(x)$$

$$=L(\underline{g},P_A)$$

Similarly $U(f,P)\ge U(\overline{g},P_A)$

So

$$L(f,P) \le L(\underline{g},P_A) \le U(\underline{g},P_A) \le U(f,P)$$

$$L(f,P) \le L(\overline{g},P_A) \le U(\overline{g},P_A) \le U(f,P)$$

if $f$ is integrable, $\forall \epsilon > 0$, find a partition $P$ s.t. $U(f,P) - L(f,P) < \epsilon$, and then

$$U(\underline{g},P_A) - L(\underline{g},P_A) < \epsilon \quad \& \quad U(\overline{g},P_A) - L(\overline{g},P_A) < \epsilon$$

So $\int \underline{g}$ & $\int \overline{g}$ both exist,

&

$$L(f,P) \le \int \overline{g} \le U(f,P)$$

$$L(f,P) \le \int \underline{g} \le U(f,P)$$

$$L(f,P) \le \int f \le U(f,P)$$

$$U(f,P) - L(f,P) < \epsilon$$

So $\left| \int \overline{g} - \int f \right| \le \epsilon$

& $\left| \underline{g} - \int f \right| \le \epsilon$

For every $\epsilon > 0$, so

$$\int f = \int \underline{g} = \int \overline{g}$$

☺

On to Partitions of Unity.

<u>Thm</u> (POI) Suppose $A \subset \mathbb{R}^n$ (not nec rect, not nec open, not nec bndd)

Suppose $\mathcal{U} = \{U\}$ is an open cover of $A$.

Then there exists a (countable) collection $\Phi = \{\varphi_i : W \to [0,1]\}$ of $C^\infty$-fnctns (meaning, that each $\varphi_i$ is diffable $\infty$-many times) on an open set $W$ containing $A$ s.t.

1. $\Phi$ is "locally finite" meaning, for each $x \in W$ there is some open set $V \ni x$, s.t. all but finitely many of the $\varphi_i$'s vanish on $V$.

2. $\forall x \in A \quad \sum_{i=1}^{\infty} \varphi_i(x) = 1$.

3. $\forall \varphi_i \in \Phi \quad \exists U \in \mathcal{U}$ s.t. $\operatorname{supp} \varphi_i \subset U$

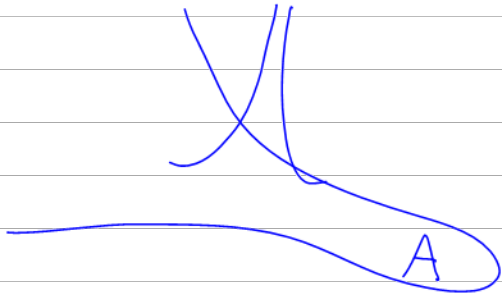
$\operatorname{supp} \varphi :=$ closure of the set $\{x : \varphi(x) \neq 0\}$

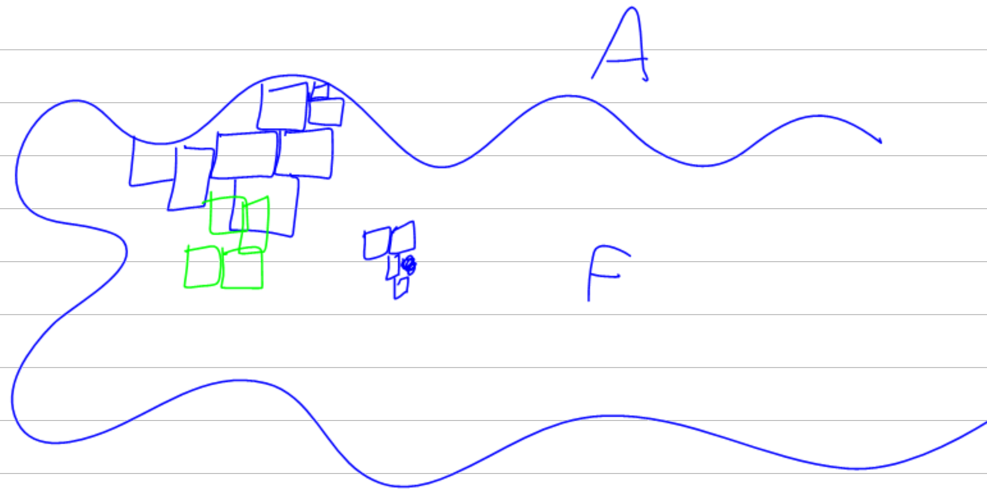
<u>Def</u> such a $\Phi$ is called a POI for $A$,

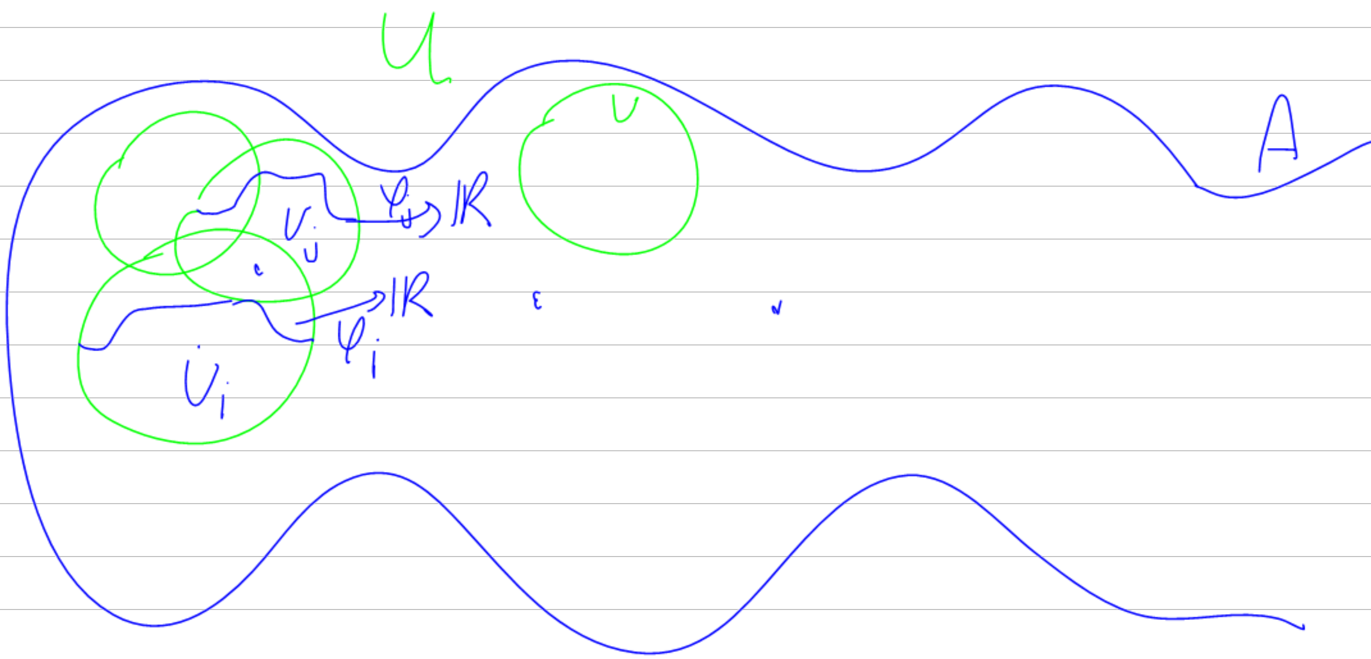
Subordinate to $\mathcal{U}$.

We wish to integrate $F : \underbrace{A}_{\text{may not be bndd}} \xrightarrow{} \mathbb{R}$, $A \subset \mathbb{R}^n$

$F$ may not be bndd




$$\forall x \in A \quad \sum \varphi_i(x) = 1$$

$$\int_A^{NT} F = \sum_i \int \varphi_i F \sim \int \left(\sum \varphi_i\right) F$$

$$= \int F$$