MAT257 Analysis II on November 2, 2020: Integrability and continuity.
Read Along: Spivak 46-56.
Read Together: https://q.utoronto.ca/courses/183900/discussion_topics/894005
Riddle Along: Cars A,B,C,D drive in the Sahara Desert on generic straight lines and at constant speed; it is known that A meets B (they arrive at the same place at the same time), A meets C, A meets D, B meets C, and B meets D. Does C necessarily meet D?

Goal: ($f: R \to \mathbb{R}$ integrable) $\Leftrightarrow$ f is cont. except on meas-0.

Thm $[a,b]$, $a<b$, does not have content 0,
(hence not measure 0, hence not countable)

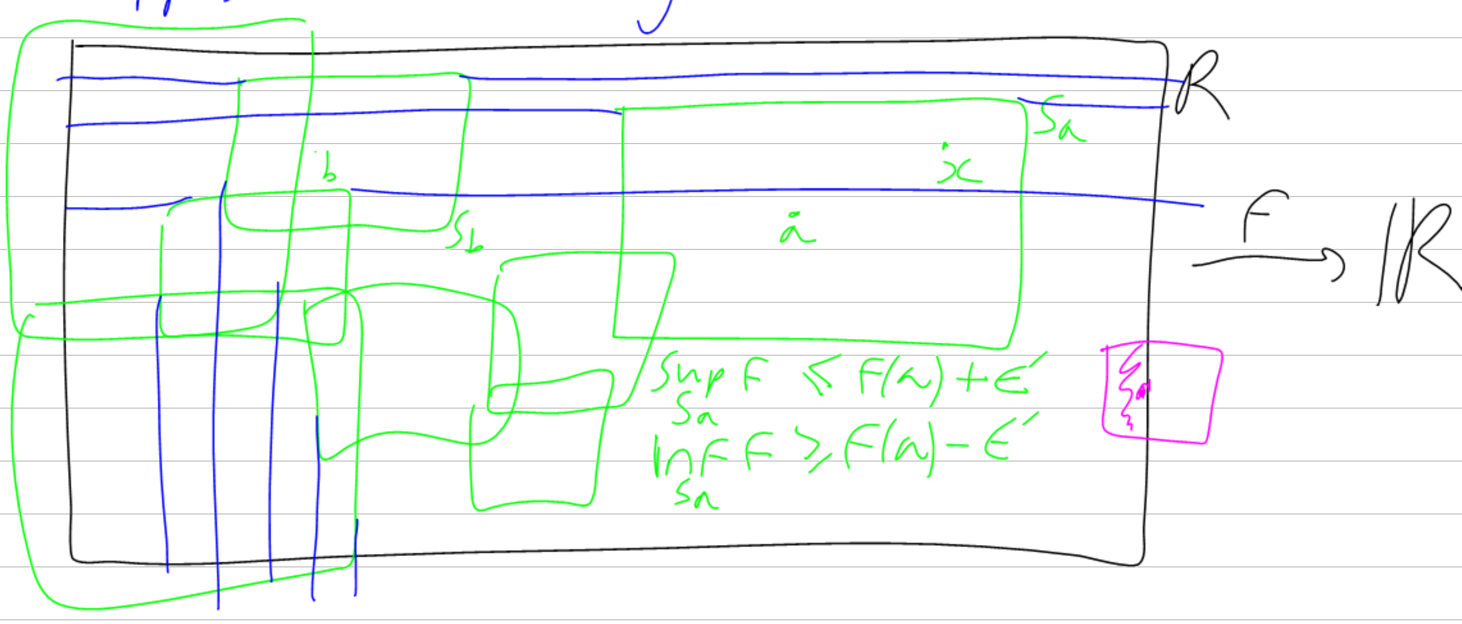
Skipped Thm $R = \prod_{i=1}^{n} [a_i, b_i]$ $\forall i\ b_i > a_i$

$\Rightarrow$ R does not have measure 0.

Thm $f: R \to \mathbb{R}$ is cont. $\Rightarrow$ f is integrable.

(f integrable) $\Leftrightarrow$ $\forall \epsilon > 0\ \exists P$ of R s.t. $U(f,P) - L(f,P) < \epsilon$

Suppose $\epsilon > 0$ is given



Pf Let $\epsilon > 0$ be given. For each $a \in R$

Use the continuity of f at a to find a closed rectangle S_a s.t. $a \in \operatorname{int} S_a$ &

$$\forall x \in S_a \cap R \quad |f(x) - f(a)| < \frac{\epsilon}{2\,\mathrm{Vol}(R)}$$

$\mathrm{Vol}(R) = V$

So $f(x) - f(a) < \frac{\epsilon}{2V}$

So $f(x) < f(a) + \frac{\epsilon}{2V}$

So $\sup_{x \in S_a \cap R} f(x) \leq f(a) + \frac{\epsilon}{2V}$

Similarly $\inf_{x \in S_a \cap R} f(x) \geq f(a) - \frac{\epsilon}{2V}$

So $\sup - \inf \leq f(a) + \frac{\epsilon}{2V} - \left(f(a) - \frac{\epsilon}{2V}\right) = \frac{\epsilon}{V}$

The collection $\{\operatorname{int} S_a\}_{a \in R}$ is an open cover of R and so we can find a finite subcover $\operatorname{int} S_{a_1}, \ldots, \operatorname{int} S_{a_k}$.

By taking all endpoints of all the S_{a_i}, get a partition P of R s.t. every $S \in P$ is a subset of one of the S_{a_i}'s.

$U(f,P) - L(f,P) =$

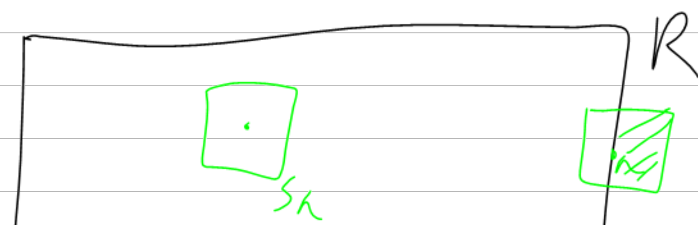
$$\sum_{S \in P} v(S) \cdot \sup_{x \in S} f(x) - \sum_{S \in P} v(S) \inf_{x \in S} f(x)$$

$$= \sum_{S \in P} v(S) \left(\sup_{x \in S} f(x) - \inf_{x \in S} f(x) \right)$$

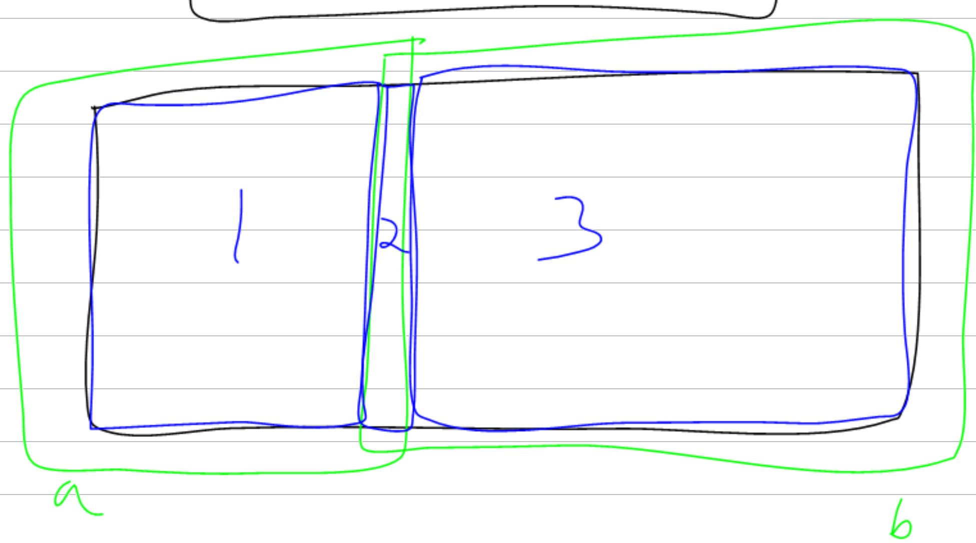
our S in contained in some S_{n_i}

$$< \sum_{S \in P} v(S) \cdot \frac{\epsilon}{v(R)} = \frac{\epsilon}{v(R)} \cdot \overbrace{\sum_{S \in P} v(S)}^{v(R)}$$

$$= \epsilon$$



$\square$



$$U^c(f,P) = \sum \text{using closed rectangles} = U(f,P)$$

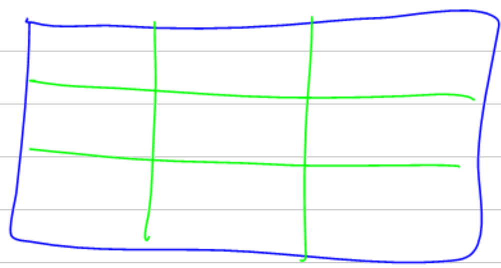
$U^o(f,P) = \sum \cdot$ using open rectangles —

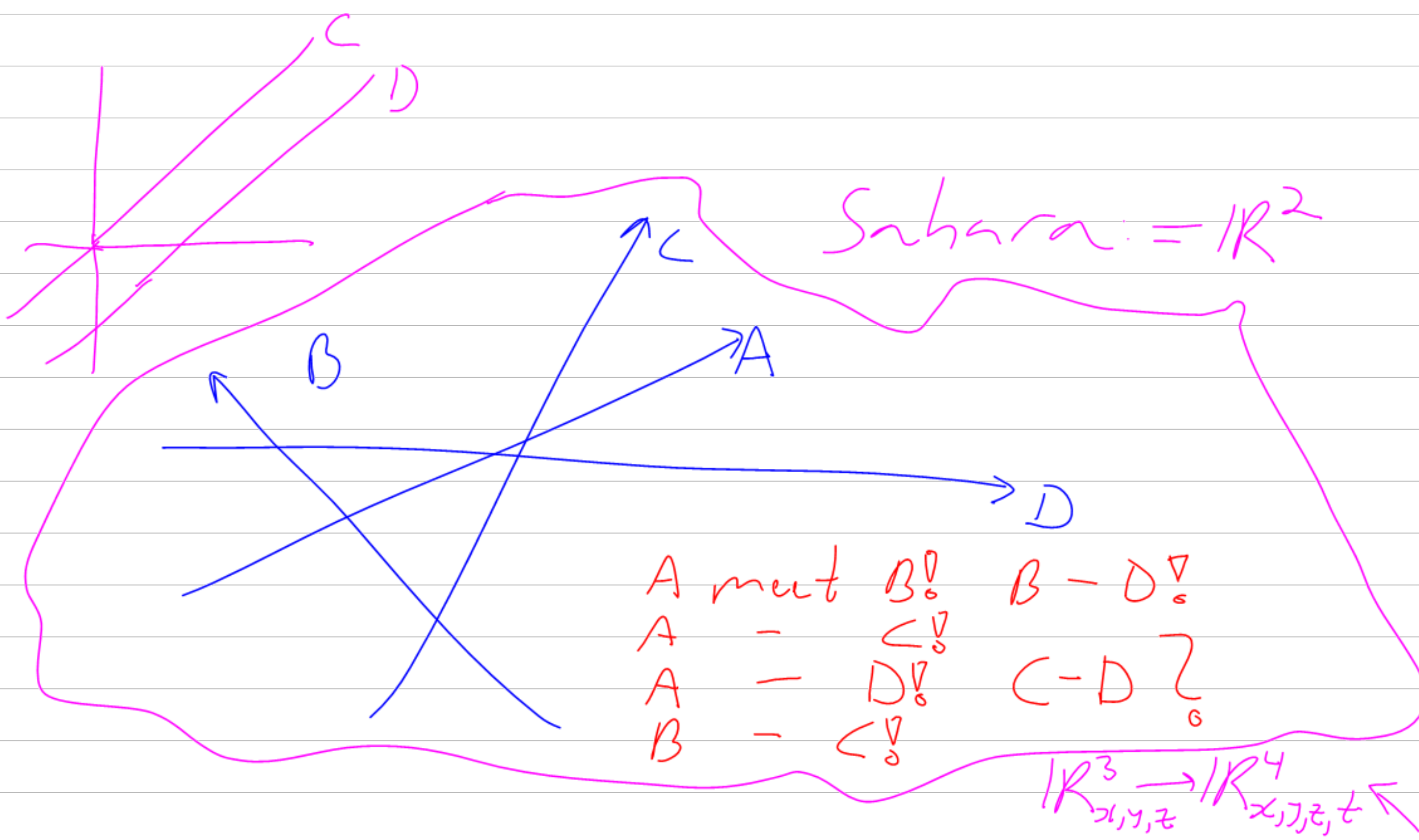
<u>Def</u> f is c-integrable if

$$U^c = L^c$$

f is o-integrable if $U^o = L^o$.

<u>Thm</u> o-integrable $\Longleftrightarrow$ c-integrable



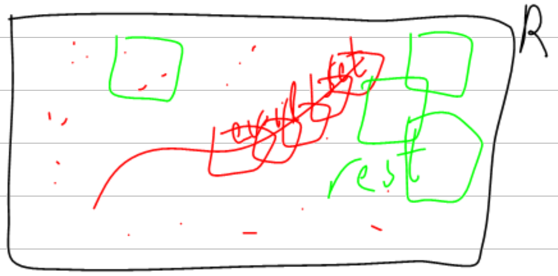


MAT257 Analysis II on November 4, 2020: Integrability, continuity, measure 0.
Read Along: Spivak 46-56 (warning: pf of Thm 3-8 is buggy; see page 145).
I hope the Term Test went well! I have no comments yet. The deadline to report technical issues is today at 7PM; several outstanding issues will be resolved tomorrow.
Riddle Along: Cars A,B,C,D drive in the Sahara Desert on generic straight lines and at constant speed; it is known that A meets B (they arrive at the same place at the same time), A meets C, A meets D, B meets C, and B meets D. Does C necessarily meet D?

Goal: ($f: R \to \mathbb{R}$ integrable) $\Leftrightarrow$ f is cont. except on meas-0.

Done ($f: R \to \mathbb{R}$ integrable) $\Leftarrow$ f is cont. everywhere.

$$U(f,P)-L(f,P)=\sum_{S\in P} v(S)\underbrace{\left(\sup_{x\in S} f(x)-\inf_{x\in S} f(x)\right)}_{\text{oscillation} \leftarrow o(f,S)}$$



$$=\sum_{S\in P} \underbrace{v(S)}_{\text{small}} \cdot \underbrace{o(f,S)}_{\text{small}}$$

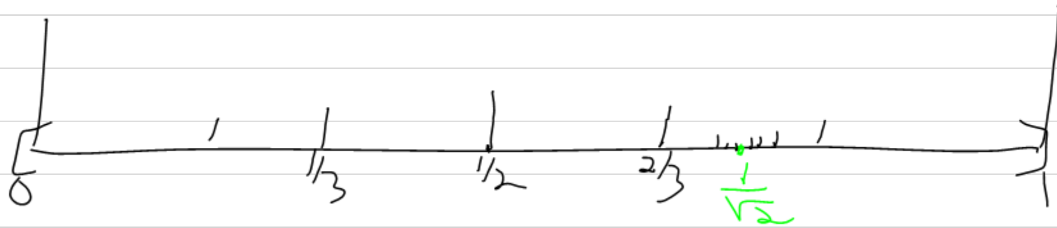
dream $< \epsilon$

Example $f:[0,1]\to\mathbb{R}$ "bed of nails function"

$$f(x)=\begin{cases}\frac{1}{q} & x=\frac{p}{q} \text{ (in lowest denom.)}\\ 0 & x\notin\mathbb{Q}\end{cases}$$

$\mathrm{disc}(f)=\mathbb{Q}$
discontinuities

$\mathrm{cont}(f)=\mathbb{Q}^c$



Def If f bndd, & $a\in R$

$$o(f,a)=\lim_{r\downarrow 0} o(f,B_r(a))$$

ball of rad r around a

The limit always exist because $o(f, B_r(a))$ is a decreasing function of r.

Thm f is cont. at a iff $o(f,a)=0$

Pf $\Longrightarrow$ If f is cont., pick $\epsilon>0$, find δ s.t. if $|x-a|<\delta$ then $|f(x)-f(a)|<\epsilon/2$.

Then $\sup_{x\in B_\delta(a)} f(x) \le f(a)+\frac{\epsilon}{2}$

$\inf_{x\in B_\delta(a)} f(x) \ge f(a)-\frac{\epsilon}{2}$

So $o(f, B_\delta(a)) \le \frac{\epsilon}{2} - \left(-\frac{\epsilon}{2}\right) = \epsilon$

So $\lim_{\delta\to 0} o(f, B_\delta(a)) = 0$

$\Longleftarrow$ Suppose $o(f,a)=0$, $\epsilon>0$,

as $o(f,a) = \lim_{\delta\to 0} o(f, B_\delta(a)) = 0$

So find $\delta>0$ s.t. $o(f, B_\delta(a)) < \epsilon$

$\inf_{B_\delta(a)} f(x) \le f(a) \le \sup_{B_\delta(a)} f(x)$

also for any $x \in B_\delta(a)$:

$\inf_B f \le f(x) \le \sup_B f(x)$

So

$$f(a), f(x) \in \left[\inf_{B_r(a)} f, \sup_{B_r(a)} f\right]$$

length of this interval is less than ϵ

So $|f(x)-f(a)| < \epsilon$

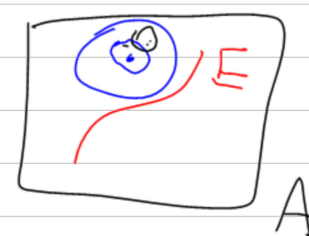
for every $x \in B_r(a)$. $\square$

<u>Aside</u> $f: A \to \mathbb{R}$ bndd, A is closed.

In this case

$$E = \{a \in A : o(f,a) \geq \epsilon\}$$

is a closed set.



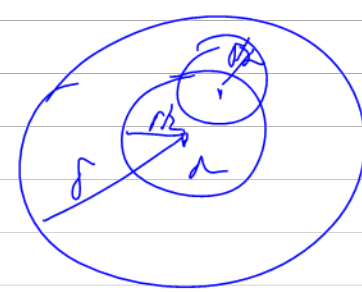
<u>Pf</u> Suppose $a \notin E$, $a \in A$,

$$\lim_{r \to 0} o(f, B_r(a)) < \epsilon$$

So $\exists\, \delta > 0$ s.t. $o(f, B_\delta(a)) < \epsilon$

Suppose $b \in B_{\delta/2}(a)$ now

$$B_{\delta/2}(b) \subset B_\delta(a)$$



$$o(f, B_{\delta/2}(b)) \leq o(f, B_\delta(a)) < \epsilon$$

So $\lim_{r \to 0} o(f, B_r(b)) < \epsilon$

So every $a \in A \cap E^c$ has a nbd $\left(B_{r/2}(a)\right)$ contained in E^c,

So E is closed. $\square$

<u>Aside</u> $\mathrm{disc}(f) = \bigcup_{n=1}^{\infty} \underbrace{\left\{a : o(f,a) \geq \frac{1}{n}\right\}}_{\text{closed}}$

So $\mathrm{disc}(f)$ is a countable union of closed sets ["F_σ set"]

<u>Riddle</u> Is it true that every F_σ set is $\mathrm{disc}(f)$ for some function f?

$\mathbb{Q}$ are F_σ

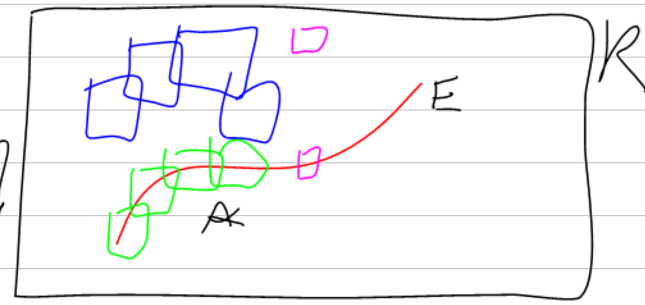
$\mathbb{Q}^c$ are not F_σ

MAT257 Analysis II on November 6, 2020: Integrability, continuity, measure 0.
Read Along: Spivak 46-56 (warning: pf of Thm 3-8 is buggy; see page 145).
Riddle Along: Cars A,B,C,D drive in the Sahara Desert on generic straight lines and at constant speed; it is known that A meets B (they arrive at the same place at the same time), A meets C, A meets D, B meets C, and B meets D. Does C necessarily meet D?

Goal: ($f: R \to \mathbb{R}$ integrable) $\Leftrightarrow$ f is cont. except on meas-0

Claim: R closed, $E \subset R$ s.t. $\forall x \in R \setminus E$ $\exists \delta > 0$ $B_\delta(x) \cap R \subset R \setminus E$
$\Rightarrow E$ is closed [For $E = R \cap \left(\bigcup\left(\begin{smallmatrix}\text{all these}\\ \text{balls}\end{smallmatrix}\right)\right)^c$]

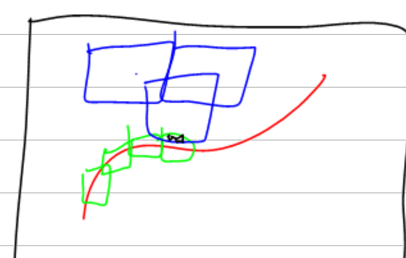
$\Leftarrow$ Given $\epsilon > 0$,



Let $E = \{a \in R : O(f,a) \geq \epsilon_1\}$
where $\epsilon_1 > 0$, TBD.
$E \subset disc(f)$, E is of meas-0. Also,
E is closed.
Cover E with a countable collection
$\mathcal{A} = \{A_i\}_{i=1}^{\infty}$ of open rectangles s.t.

$$\sum_{i=1}^{\infty} v(A_i) < \epsilon_2 \qquad (\epsilon_2 \text{ is TBD})$$

For each $a \in R \setminus E$,

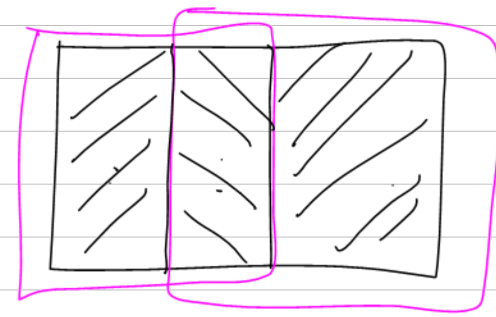


we know $o(f,a) < \epsilon_1$
Find an open rectangle $B_a \ni a$
s.t. $O(f, B_a) < \epsilon_1$

Let $\mathcal{B} = \{B_a : a \in R \setminus E\}$ covers while $\mathcal{A}$ covers E. $R\setminus E$.

So $\mathcal{A}\cup\mathcal{B}$ covers R, Using compactness, find some finite subcover $\mathcal{C}\subset\mathcal{A}\cup\mathcal{B}$.

Let P be a partition of R s.t. every $C\in\mathcal{C}$ is a union of $S\in P$.



$$U(f,P)-L(f,P)=\sum_{S\in P} V(S)\cdot O(f,S)$$

$$\le \sum_{\substack{S\in P\\ \exists i\ S\subset A_i}} V(S)\cdot \underbrace{O(f,S)}_{\le 2M} + \sum_{\substack{S\in P\\ \exists B\in\mathcal{B}\ S\subset B}} V(S)\cdot \underbrace{O(f,S)}_{\le \epsilon_1}$$

$$< 2M\cdot\sum_{\substack{S\in P\\ S\subset A_i}} V(S) + \epsilon_1\cdot\sum_{\substack{S\in P\\ S\subset B}} V(S)$$

If $|f|\le M$ on R, then $O(f,S)\le 2M$

$$< \underbrace{2M\cdot\epsilon_2}_{\frac{1}{3}\epsilon} + \underbrace{\epsilon_1 V(R)}_{\frac{1}{3}\epsilon} < \epsilon$$

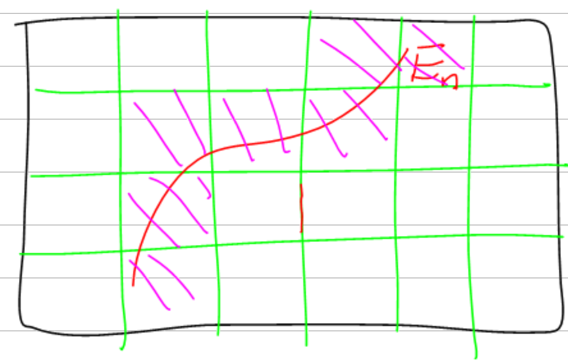
when $\epsilon_2=\frac{\epsilon}{6M}$ & $\epsilon_1=\frac{\epsilon}{3V(R)}$ $\square$

$\Longrightarrow$ Suppose f is integrable.

$$\mathrm{disc}(f) = \{a : O(f,a) > 0\}$$

$$= \bigcup_{n=1}^{\infty} \underbrace{\{a : O(f,a) \geq \tfrac{1}{n}\}}_{E_n}$$

Enough to show that E_n is of meas-0, because a countable union of meas-0 sets is meas-0.



Let $\epsilon > 0$ be given

Find a partition P of R s.t.

$$\sum_{S \in P} v(S) \cdot O(f,S) < \epsilon_1 \quad \text{(TBD)}$$

$\boxed{\text{choose } \epsilon_1 < \frac{\epsilon}{n}.}$

$$\Downarrow$$

$$\frac{1}{n} \sum_{\substack{S \in P \\ (\mathrm{int}\, S) \cap E_n \neq \emptyset}} v(S) \Longrightarrow \underbrace{\sum_{\substack{S \in P \\ (\mathrm{int}\, S) \cap E_n \neq \emptyset}} v(S)}_{\text{covers } E_n,} < n\epsilon_1$$

$$< \epsilon$$

except perhaps for $\underbrace{\bigcup_{S\in P} \mathrm{bd}(S)}_{\text{in itself of meas-0,}}$

So overall E_n is of meas-0.

$\square$