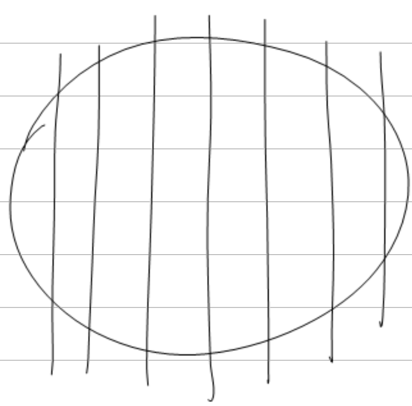


$\mathbb{Q} \rightsquigarrow \mathbb{R}$

Can you find uncountably many subsets of $\mathbb{Z}$ s.t.

A. The intersection of any two is finite?

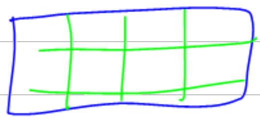
B. For any two, one includes the other?

3, 3.1, 3.14, 3.141 . . .

MAT257 Analysis II on October 26, 2020: The definition of the integral.
Read Along: Spivak 46-56.
TT1 discussion on Wednesday. Until then, just keep on top of things!
Riddle Along: A spherical loaf of bread is put into a bread-cutting machine. Which slice gets the most crust?

$$\boxed{\int_M dw = \int_{\partial M} w}$$

$R = \prod [a_i, b_i]$ $\quad P: (P_i)$ where P_i partitions $[a_i, b_i]$



$P_i = (a_i = t_{i0} \leq \ldots \leq t_{iN_i} = b_i)$

R is divided into a union of nearly-disjoint subrectangles $\prod_{i=1}^{n} [t_{i,j_i - 1}, t_{i j_i}]$ $\prod N_i$ of them.

$v(R) := \prod (b_i - a_i)$ $\quad$ __Claim__ $\quad v(R) = \sum_{S \in P} v(S)$

$f : R \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ bndd.

$f_1 \equiv c$

$$f_2(x) = \begin{cases} 1 & \forall i \; x_i \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Given a partition P of R, and $S \in P$

$m_S(f) = \inf_{x \in S} f(x) \quad m_S(f_1) = c \quad m_S(f_2) = 0$ [1]

$M_S(f) = \sup_{x \in S} f(x) \quad M_S(f_1) = c \quad M_S(f_2) = 1$ [1]

[1] unless S is degenerate $v(S) = 0$.

__Def__ Lower sum for f rel. a partition P

$$L(f, P) = \sum_{S \in P} v(S) \cdot m_S(f)$$

upper sum... $U(f,P) = \sum v(S) M_S(f)$

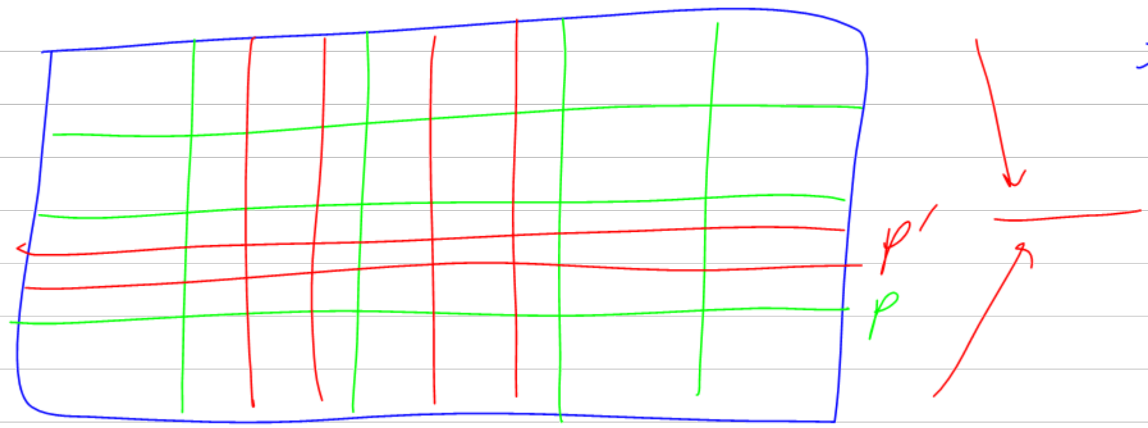
$$L(f_1,P) = \sum_S V(S)\cdot c = c\cdot \mathrm{Vol}(R)$$

$$U(f_1,P) = \cdots\cdots$$

$$L(f_2,P) = \sum_S V(S)\cdot m_S(f) = 0$$

$$U(f_2,P) = \sum_S V(S)\cdot M_S(f) = \sum V(S)\cdot 1 = V(R)$$

<u>Claim</u> $L(f,P) \leq U(f,P)$ [Pf: Follows from $m_S(f) \leq M_S(f)$; $\leq f(x)$, $x \in S$]



<u>Def</u> If P & P' are partitions of $[a,b]$

$(t_0 \leq \dots \leq t_n)$ $\quad (t_0' \dots\dots t_{n'}')$

We say that P' <u>refines</u> P if for every t_i there is some i' s.t. $t_i = t_{i'}'$

If P & P' are partitions of $R \subset \mathbb{R}^n$

$(P_i)_{i=1}^n$ $\quad (P_i')$

Say that P' refines P if $\forall i$ P_i' refines P_i

<u>Claim</u> If P' refines P $\quad$ [$R \subset \mathbb{R}^n$, $f: R \to \mathbb{R}$]

thm $$L(P,F) \overset{?}{\leq} L(P',F) \overset{\checkmark}{\leq} U(P',F) \overset{?}{\leq} U(P,F)$$

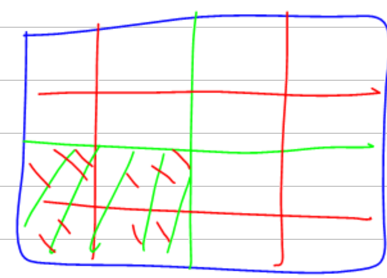
<u>Pf</u>

$$L(P,F) = \sum_{S\in P} v(S)\cdot m_S(F)$$

$$= \sum_{S\in P}\left(\sum_{\substack{S'\in P'\\ S'\subset S}} v(S')\cdot m_S(F)\right)$$

$m_S(F) \leq m_{S'}(F)$ always true

$$\leq \sum_{S\in P}\ \sum_{\substack{S'\in P'\\ S'\subset S}} v(S')\, m_{S'}(F)$$

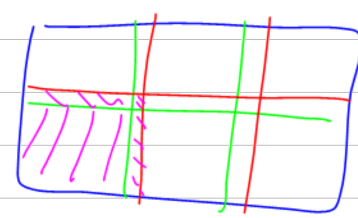


$$= \sum_{S'\in P'} v(S')\, m_{S'}(F) = L(F,P')$$

$\square$

<u>Claim</u> If P & P' are any two parti...

then
$$L(F,P) \leq U(F,P')$$



<u>Pf</u> Find a partition P'' that refines both.

$$P''_i = \left(\{t_{ij}\}_{j=1}^{N_i} \cup \{t'_{ij'}\}_{j=1}^{N_i'} \text{ ordered}\right)$$

$$L(f,P) \leq L(f,P'') \leq U(f,P'') \leq U(f,P') \quad \square$$

<u>Def</u>

lower integral of f

$$L(f) = \sup_P L(f,P) \qquad \left(\text{easy: } L(f) \leq U(f)\right)$$

$$U(f) = \inf_P U(f,P)$$

Say that "f is integrable on R" if $L(f) = U(f)$

and in this case we define

$$\int_R f = L(f) = U(f).$$

$$f_1 = c \qquad f_2 = \begin{matrix} 1 \\ 0 \end{matrix} \qquad \mathbb{Q}$$

$$L(f_1) = \sup(L(f_1,P)) = \sup\{c \cdot vol(R)\}$$
$$= c \cdot vol(R)$$

$$U(f_1) = \ldots = c \cdot vol(R)$$

$$\Rightarrow f_1 \text{ is integrable } \& \int_R f_1 = c \cdot vol(R)$$

$$L(F_2) = \sup(0) = 0$$

$$V(F_2) = \inf\{\mathrm{Vol}(R)\} = \mathrm{Vol}(R)$$

unless R is flat, F_2 is not integrable

$$\cancel{\int_R F_2} \qquad \mathrm{Volume}(R)$$

MAT257 Analysis II on October 28, 2020: Integrability and continuity.
Read Along: Spivak 46-56.
Riddle Along: Your turn!

On TT1:
* Tuesday November 3, 5-7PM (Toronto time), on Crowdmark. Other than documented accessibility matters, no exceptions!
* I will be available to answer questions throughout the exam, at my usual office (http://drorbn.net/vchat, but I'll add a waiting room).
* There will be mishaps! I just hope that not too many. If you encounter one, document everything with specific details, times, and screen shots, and send me a message by Wednesday November 4 at 7PM. I will deal with these situations on a case by case basis.
* Material: Everything up to but not including integration. [start, integration)
* Open book(s), open notes, but you can only use the internet (during the exam) to read the exam, to submit the exam, and to connect with the instructor/TAs to ask clarification questions. No contact allowed with other students or with any external advisors, online or in person.
* "Solve 8 of 8", or maybe "10 of 10", or "7 of 7".
* You will be required to copy in your handwriting and sign an academic integrity statement and submit it to Crowdmark along with the rest of your exam. You will be given an extra 15 minutes for this purpose.
* The vast majority of students will do honest work, and I appreciate that. Out of respect for the honest students I will do my best to pursue and punish any cheating that may occur. I'm more experienced than you! If you plan to be dishonest, think again.
* To prepare: Do the TT1 "rejects", but more important: make sure that you understand every single bit of class material so far!
* It is not the exam I want! Class material and HW are important, but there won't be questions straight from class/HW. Many things in 2020 are not as we want them.

Corollary If P & P' are any two partitions.

$$L(f,P) \le U(f,P')$$

Def $U(f) := \inf_P U(f,P)$ $\quad L(f) = \sup_P L(f,P)$

Def integrable $\iff U(f)=L(f)$

and then $\int_R f = U(f) = L(f)$.

$\boxed{f: R\subset\mathbb{R}^n \to \mathbb{R} \text{ bdd}}$

Thm f is integrable $\iff \forall \epsilon>0\ \exists P$ s.t.

$$U(f,P) - L(f,P) < \epsilon$$

Pf $\Rightarrow$ if f is integrable, $U(f)=L(f)=I$

by def $\exists$ some P' s.t.

$$U(f,P') < U(f) + \epsilon/2$$

likewise there is some P'' s.t.

$$L(f,P'') > L(f) - \epsilon/2$$

Let P be a refinement of both P' & P''

then $U(f,P)-L(f,P)$

$$\leq U(f,P')-L(f,P'') \leq \underline{U}(f)+\frac{\epsilon}{2}-\left(\underline{L}(f)-\frac{\epsilon}{2}\right)$$

$$= \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon \quad \checkmark$$

$\Leftarrow$ Suppose for every ϵ

$\exists P$ s.t. $U(f,P)-L(f,P) < \epsilon$

$\Rightarrow 0 \leq U(f)-L(f) < \epsilon$

$\Rightarrow U(f)=L(f) \Rightarrow f$ is integrable.

<u>Goal</u> $f: R \to \mathbb{R}$ is integrable iff it is continuous except on a <u>tiny</u> set

<u>Def</u> A set $A \subset \mathbb{R}^n$ is of measure 0 ("n-dimensional measure 0")

Formal name: Sets of measure 0, measure-0 sets

If for every $\epsilon > 0$ you can find countably many open rectangles R_i s.t.

1. $A \subset \bigcup_{i=1}^{\infty} R_i$ 2. $\sum_{i=1}^{\infty} v(R_i) < \epsilon$

<u>Examples</u> 1. Every finite set is of meas-0.

$A = \{a_1 \dots a_n\}$

Given $\epsilon > 0$, find open rectangles $R_1 \dots R_n$ s.t.

$$a_i \in R_i \quad v(R_i) < 2^{-i}\epsilon$$

add open rectangles R_i anywhere, provided $v(R_i) < 2^{-i}\epsilon$, for $i > n$.

Clearly $A \subset \bigcup R_i$

and

$$\sum v(R_i) < \sum_{i=1}^{\infty} 2^{-i}\epsilon = \overset{\overset{\epsilon}{\|}}{\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)\epsilon}$$

<u>Example 2</u> If A is countable, meaning $A = \{a_i\}_{i=1}^{\infty}$, then A is of meas-0.

<u>Pf</u> Given $\epsilon > 0$, choose R_i s.t.

$$a_i \in R_i \ \& \ v(R_i) < \epsilon \cdot 2^{-i}$$

cont. as before.

<u>Aside</u> $\mathbb{R}$ (likewise $\mathbb{R}^n$ for $n > 0$) is not countable!

You cannot find a sequence $\{a_i\}$ of real numbers that goes through all of $\mathbb{R}$,

$$\{a_i\} = \mathbb{R}$$

Pf: "Cantor diagonalization" assume $\{a_i\}$ existed, write the a_i's in decimal:

$$a_1 = \sim\sim\sim . d_{11} d_{12} d_{13} \dots$$
$$a_2 = \sim\sim\sim . d_{21} d_{22} d_{23} \dots$$
$$a_3 = \sim\sim\sim . d_{31} d_{32} d_{33} \dots$$

Define

$$x = 0.e_1 e_2 e_3 \dots$$

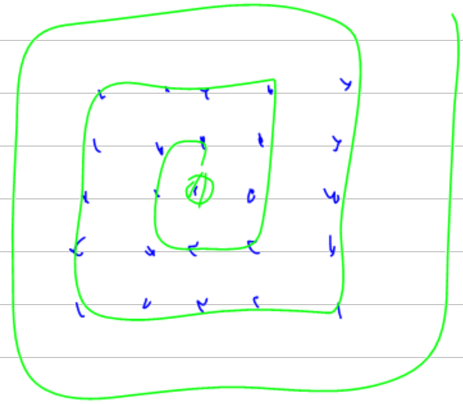
$$e_i = \begin{cases} 3 & \text{if } d_{ii} \neq 3 \\ 7 & \text{if } d_{ii} = 3 \end{cases}$$

Clearly $x \neq a_i$ for all i, so

$$x \notin \{a_i\} \quad \text{so} \quad \{a_i\} \neq \mathbb{R}.$$

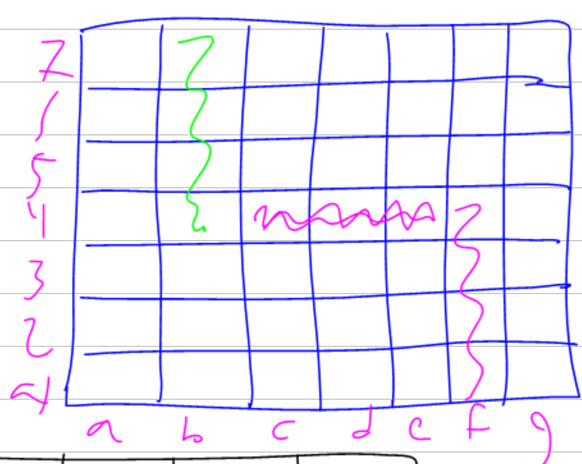
Example $\mathbb{Q}$ are countable hence of meas 0.

$$\mathbb{Q} = \left\{ \frac{a}{b} ;\ a, b \in \mathbb{Z} \right\}$$



$$17 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{0}}}$$

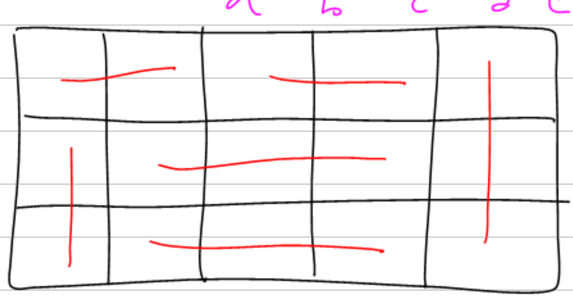
$\mathbb{R}^{\cancel{\#}} \sim N^N$

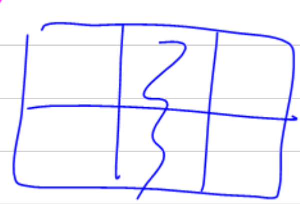


2×1

3×1

4×1





MAT257 Analysis II on October 30, 2020: Measure 0 and content 0, integrability and continuity.
Read Along: Spivak 46-56.
Riddle Along: Players A and B alternate placing 1x2, 1x3, and 1x4 lego pieces (as they choose) on a 19 x 21 board, with no layering and no overlaps. If you cannot place a piece, you lose. Who would you rather be A or B? What if the overall size was 20 x 20?

Goal: ($f: \mathbb{R} \to \mathbb{R}$ integrable in $\mathbb{R}^n$) $\iff$ (f is cont. except on a set of measure 0)

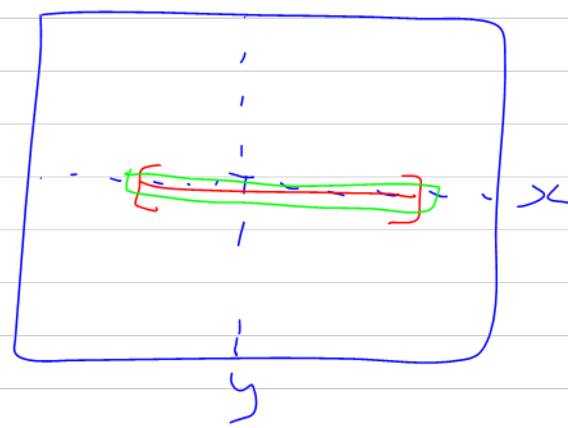
Def A is meas-0 means $\forall \epsilon > 0$ $\exists$ open rectangles (R_i) s.t. (could be closed.)

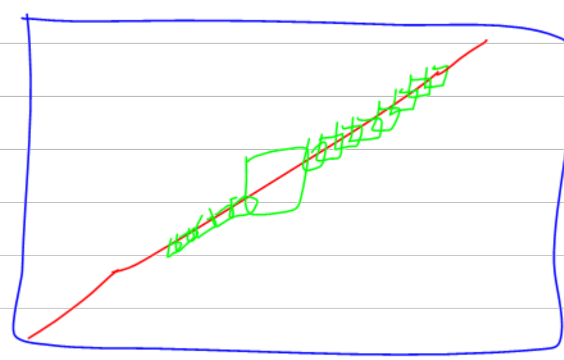
1. $A \subset \bigcup R_i$ 2. $\sum v(R_i) < \epsilon$

Example Finite & countable sets are meas-0.

Example 1. $[a,b] \subset \mathbb{R}^2$

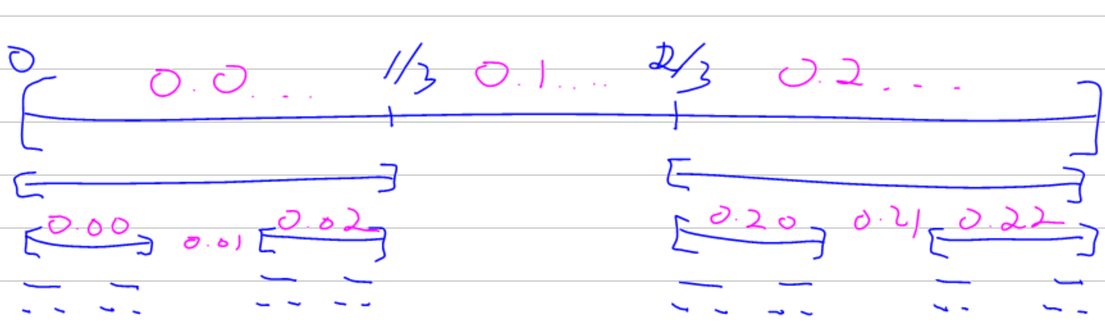
2. any straight line in $\mathbb{R}^n$, $n > 1$ is of meas-0.





Example Cantor set: $C \subset [0,1] \subset \mathbb{R}$:

0.01211003



$C_1 = [0,1]$

$C_2 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$

C_3

C_4

C_5

$$C = \bigcap_{n=1}^{\infty} \underset{\substack{\uparrow \\ \text{closed}}}{C_n} = \left\{ x \overset{[0,1]}{\in} : \text{ the trinary expansion of } x \text{ has only } 0\text{s} \,\&\, 2\text{s} \right\}$$

* C is closed hence compact.

* C is uncountable.

* C is of meas 0.

0

0.200222002

0.00202002

0.2222002

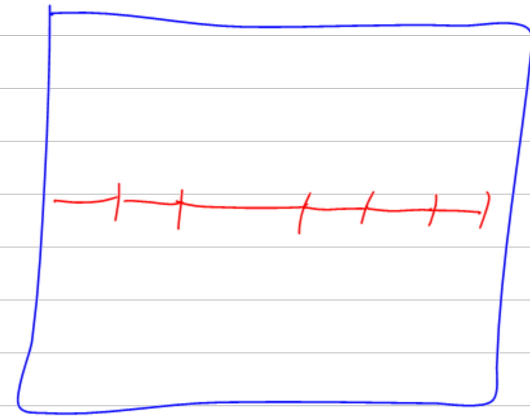
C is covered by C_n, which is a union of 2^{n-1} intervals each of length $\left(\frac{1}{3}\right)^{n-1}$, so C is covered by intervals whose sum of volumes is

$$2^{n-1}\left(\frac{1}{3}\right)^{n-1} = \left(\frac{2}{3}\right)^{n-1} \xrightarrow[n\to\infty]{} 0$$

<u>Fact 1</u> A subset of a meas-0 set is meas-0.

<u>Fact 2</u> A countable union of meas-0 is meas-0. So if A_i is meas-0 for all $i \in \mathbb{N}$, then $A = \bigcup_{i=1}^{\infty} A_i$ is

also meas 0.



Pf Given $\epsilon > 0$ for each i, find a cover $(U_{ij})_{j=1}^{\infty}$ of A_i s.t. $\sum_{j=1}^{\infty} v(U_{ij}) < 2^{-i}\epsilon$

↑ an open rectangle. Define R_k as follows

$$\begin{array}{lllll} R_1 = U_{11} & R_2 = U_{12} & R_4 = U_{13} & R_7 = U_{14} & \cdots \\ R_3 = U_{21} & R_5 = U_{22} & U_{23} & \cdots & \\ R_6 = U_{31} & U_{32} & \cdots & & \end{array}$$

$$\bigcup R_k = \bigcup U_{ij} \supset \bigcup A_i \supset A$$

$$\sum v(R_k) = \sum_{i,j} v(U_{ij}) = \sum_i \sum_j v(U_{ij})$$

$$< \sum_i 2^{-i}\epsilon = \epsilon$$

Thm $[a,b]$, $a<b$ is not of meas-0 in $\mathbb{R}^1$

__Lemma__ If you cover $[a,b]$ with n (finite) open intervals (α_i,β_i) $i=1,\dots,n$. Then

$$\sum v((\alpha_i,\beta_i)) \geq b-a$$

Sketch induction

$\Longrightarrow$ Even if $\bigcup_{i=1}^{\infty}(\alpha_i,\beta_i)$ covers $[a,b]$, then $\displaystyle\sum_{i=1}^{\infty} v((\alpha_i,\beta_i)) \geq b-a$.

__Pf__ Use compactness.

$\Longrightarrow$ $[a,b]$ is not meas-0. $\square$