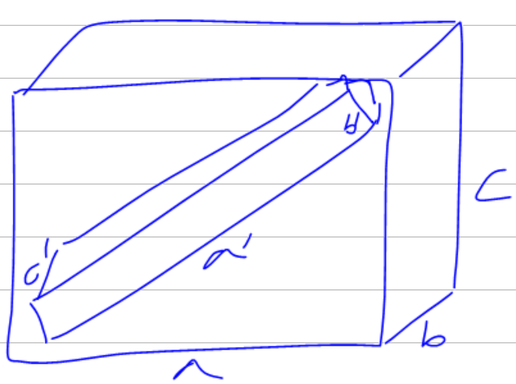


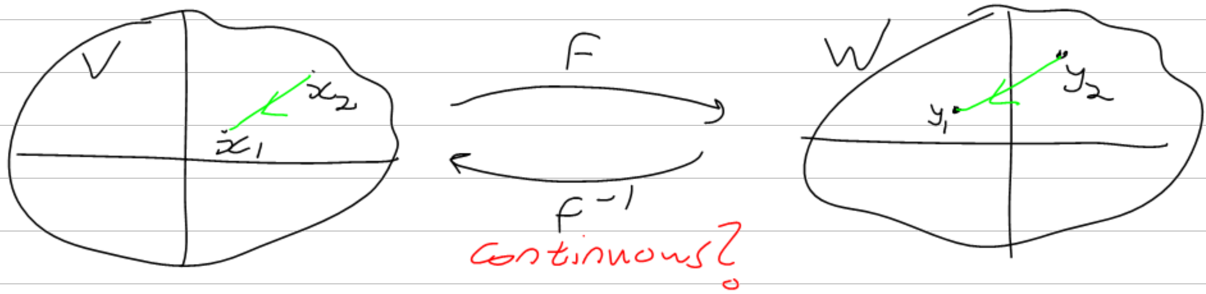
"ghosting"?

$a'+b'+c' > a+b+c$?

$a+b+c \leq 150$

MAT257 Analysis II on October 19, 2020: The Inverse/Implicit Function Theorems.
Read Along: Spivak 34-45.
Riddle Along: On any pair of potatos, can you draw a pair of 3D congruent curves?

$$y_i = F(x_i) \qquad x_i = F^{-1}(y_i) \qquad i=1,2$$



ASF:

$$|\underbrace{(x_1-x_2)}_{\alpha}-\underbrace{(y_1-y_2)}_{\beta}| \le \frac{1}{257}|\underbrace{x_1-x_2}_{\alpha}|$$

Trouble!

$$|\alpha-\beta| \le \frac{1}{257}|\alpha| = \frac{1}{257}|\beta+(\alpha-\beta)| \le \frac{1}{257}\left(|\beta|+|\alpha-\beta|\right)$$

$$\Rightarrow \frac{256}{257}|\alpha-\beta| \le \frac{1}{257}|\beta| \Rightarrow |\alpha-\beta| \le \frac{1}{256}|\beta|$$

$$|(x_1-x_2)-(y_1-y_2)| \le \frac{1}{256}|y_1-y_2|$$

green stuff $\rightarrow$ $\le$

$$|x_1-x_2|-|y_1-y_2|$$

ASF for F^{-1}

$\alpha = \beta + (\alpha-\beta)$

$|\alpha| \le |\beta| + |\alpha-\beta|$

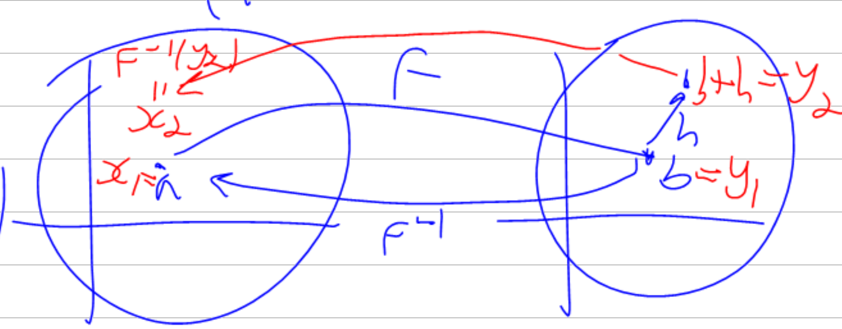
$|\alpha| - |\beta| \le |\alpha-\beta|$

$$|x_1-x_2| \le \frac{257}{256}|y_1-y_2|$$

$\Rightarrow$ cont. of F^{-1}

Diffability of F^{-1} at b:

$$F^{-1}(b+h) = F^{-1}(b) + I\cdot h + e(h)$$



$$x_2 = x_1 + y_2 - y_1 + e(h)$$

$$|e(h)| = |(x_2 - x_1) - (y_2 - y_1)| \leqslant \frac{1}{256}|y_2 - y_1|$$
$$= \frac{1}{256}|h|$$

$$\Longrightarrow \frac{|e(h)|}{|h|} \leqslant \frac{1}{256}$$

By the same reasoning, in an even smaller nbd of b, we'd have

$$\frac{e(h)}{|h|} \leqslant \frac{1}{1350}$$

...... so we can make it as small as we wish, so

$$\frac{|e(h)|}{|h|} \longrightarrow 0.$$

So $e(h) \in o(h)$, so f^{-1} is diffable at b.

Why is f^{-1} diffable away from b?

<u>Ans</u>: If the conditions for the IFT hold at b, they hold near b.

* $D_i f_j$ cont.
* $f'(x)$ is invertible at $x = a$.

$\Updownarrow$

$\det(f'(x)) \neq 0$.

a cont. fn of x.

$\Rightarrow \det(F'(x)) \neq 0$ also near $x=a$

$\Rightarrow F'(x)$ is invertible near $x=a$.

Rerun the whole pf & find that F^{-1} is diffable on images of pts near a, meaning at pts near b.

Why is $F^{-1}(y)$ <u>cont.</u> diffable near b?

By chain rule,

$$(F^{-1})'(y) = \left[F'(F^{-1}(y))\right]^{-1}$$

is cont ∇

$F^{-1}(y)$ is cont. in y

$F'(x)$ is cont in x

$M \mapsto M^{-1}$ is a cont. op. on matrices

$\mathbb{R}^{n^2} \xrightarrow{\text{inv}} \mathbb{R}^{n^2}$ where defined, inv is cont. by Cramer's law.

L linear trans. $\Rightarrow$ L^{-1} is a lin trans. $\Rightarrow$ L^{-1} is cont.

$\square$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \longmapsto \begin{pmatrix} \frac{d}{\det} \\ \frac{-b}{\det} \\ \vdots \\ \vdots \end{pmatrix}$$

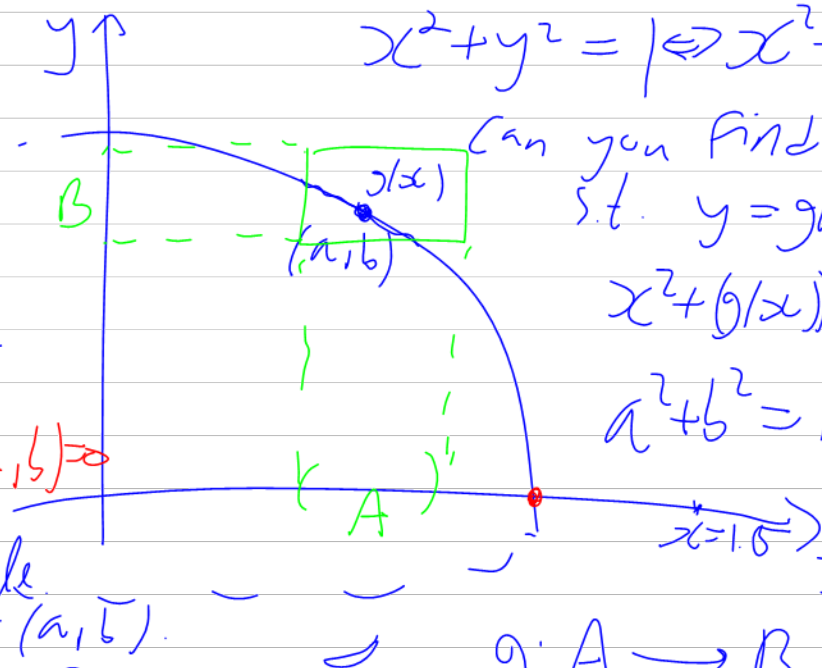
MAT257 Analysis II on October 21, 2020: The Implicit Function Theorem, Integration.
Read Along: Spivak 40-45. Read Ahead: Spivak 46-52, "Integration".
HW5 due at midnight, HW6 is on web.
Riddle Along: Can you find uncountably many sets of natural numbers, such that the intersection of any two of them is finite?

$x^2+y^2=e^{x+y}$

$\iff x^2+y^2-e^{x+y}=0$

$y=f(x)$

$x^2+y^2=1 \iff x^2+y^2-1=0$



Can you find $g(x)$ s.t. $y=g(x)$

$x^2+(g(x))^2=1$

$a^2+b^2=1$

$g: A \to B$

Thm Given $F: \mathbb{R}^n_{x_1 \dots x_n} \times \mathbb{R}^k_{y_1 \dots y_k} \to \mathbb{R}^k$ cont. diffable near $(a,b) \in \mathbb{R}^n \times \mathbb{R}^k$, $F(a,b)=0$ and $\partial F/\partial y$ is invertible at (a,b).

Then $\exists$ a nbd A of a, a nbd B of b, $\exists!$ $g: A \to B$ s.t. $g(a)=b$ & $\forall z \in A$ $F(z, g(z))=0$.

Furthermore, g is cont. diffable, and

$g' = \underline{\qquad\qquad\qquad}$

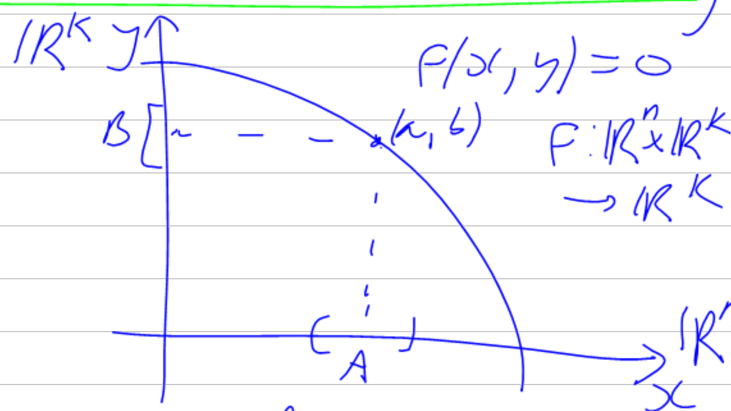
Summary

Given x solve for y.

Ahmed $\xrightarrow{F(x,y)}$ Betty

Betty: I want 0!

Ahmed can cheat! okay, take (a,b).



$F(x,y)=0$

$F: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$

$g: A \to B$ s.t. $F(x, g(x))=0$

$$(x,y) \longmapsto (x, F(x,y))$$

Betty: I want $(x,0)$

<u>PF</u> Define $H: \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n \times \mathbb{R}^k$

only near (a,b), by

$$H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ F(x,y) \end{pmatrix}$$

plan

$$g(x) = \pi_2\left(H^{-1}\begin{pmatrix} x \\ 0 \end{pmatrix}\right)$$

Reminder

$$\phi : \mathbb{R}^n_{x_1 \ldots x_n} \to \mathbb{R}^m$$

$$\phi' = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \phi_m}{\partial x_1} & & \frac{\partial \phi_m}{\partial x_n} \end{pmatrix}$$

Check cond. of IFT for H:

$$H(a,b) = \begin{pmatrix} a \\ F(a,b) \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

H is cont. diff'ble near (a,b)

$$H'_{(a,b)} = \left.\begin{array}{c|cc} & n & k \\ \hline n & \frac{\partial x_j}{\partial x_i} & \frac{\partial x_j}{\partial y_i} \\ k & \frac{\partial F_j(x,y)}{\partial x_i} & \frac{\partial F_j}{\partial y_i} \end{array}\right|_{\text{at } a,b} = \left(\begin{array}{c|c} I & 0 \\ \hline \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{array}\right)$$

invertible $\Leftrightarrow \frac{\partial F}{\partial y}$ is invertible, as given in the theorem.

So H^{-1} exist and is cont. diff'ble in a nbd of $\begin{pmatrix} a \\ 0 \end{pmatrix}$ so set

$$g(x) = \pi_2\left(H^{-1}\begin{pmatrix} x \\ 0 \end{pmatrix}\right)$$

g is defined on some open set A containing a

$$g(a) = \pi_2\left(H^{-1}\begin{pmatrix} a \\ 0 \end{pmatrix}\right) = \pi_2\begin{pmatrix} a \\ b \end{pmatrix} = b$$

So $g : A \to B$ where B is a nbd of b.

$$F(x, g(x)) = F\left(x, \pi_2 H^{-1}\begin{pmatrix} x \\ 0 \end{pmatrix}\right) = \#$$

recall, $H(x,y) = (x, F(x,y))$

So $H^{-1}(x,0) = (x, \overset{\text{some}}{y})$ s.t.

$$F(x,y) = 0.$$

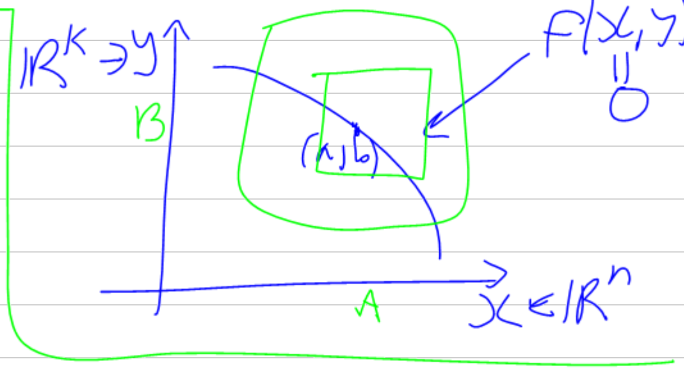
$$\# = F(x, \pi_2(x,y)) = F(x,y) = 0$$

Find g': $\quad g : x \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix} \xrightarrow{H^{-1}} \begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{\pi_2} y$

$$g' = {}_n\underset{n+k}{\begin{pmatrix} 0 & 1 \end{pmatrix}} \begin{pmatrix} I & 0 \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix}^{-1} \begin{array}{c} n \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} n \\ k \end{array} = \ldots$$

MAT257 Analysis II on October 23, 2020: The Implicit Function Theorem (a bit), and integration (boot up).
Read Along: Spivak 40-52.
TT1 discussion on Wednesday October 29. Until then, just keep on top of things!
Riddle Along: Can you find uncountably many sets of natural numbers, such that for any two of them, one contains the other?

Thm Given $f:\mathbb{R}^n_{x_1\dots x_n}\times\mathbb{R}^k_{y_1\dots y_k}\to\mathbb{R}^k$ cont. diff'able near $(a,b)\in\mathbb{R}^n\times\mathbb{R}^k$ and s.t. $f(a,b)=0$ and $\frac{\partial f}{\partial y}\in M_{k\times k}(\mathbb{R})$ is invertible, $\exists$ nbd A of a, nbd B of b, & $\exists \underline{\underline{!}}\, g:A\to B$ s.t. $g(a)=b$ & $\forall z\in A$ $f(z,g(z))=0$. Furthermore, g is cont. diff'able & $\underline{\underline{g'=-\left(\frac{\partial f}{\partial y}\right)^{-1}\frac{\partial f}{\partial x}}}$.



$$F(z,y)=0 \iff \begin{matrix} x=z \\ f(x,y)=0\end{matrix}$$

w/ $H(x,y)=\begin{pmatrix}x\\ f(x,y)\end{pmatrix}$

$$\iff H(x,y)=\begin{pmatrix}z\\0\end{pmatrix}$$

$$\iff \begin{pmatrix}x\\y\end{pmatrix}=H^{-1}\begin{pmatrix}z\\0\end{pmatrix}$$

$$\iff y=\pi_2 H^{-1}\begin{pmatrix}z\\0\end{pmatrix}$$

Set $g(z)=\pi_2\left(H^{-1}\begin{pmatrix}z\\0\end{pmatrix}\right)$

$$0=f(x,g(x)) \qquad \mathbb{R}^n\longrightarrow\mathbb{R}^k$$

So

$$0=\underset{k\times n}{\frac{\partial f}{\partial x}}+\underset{k\times k}{\frac{\partial f}{\partial y}}\cdot\underset{k\times n}{\frac{\partial g}{\partial x}}$$

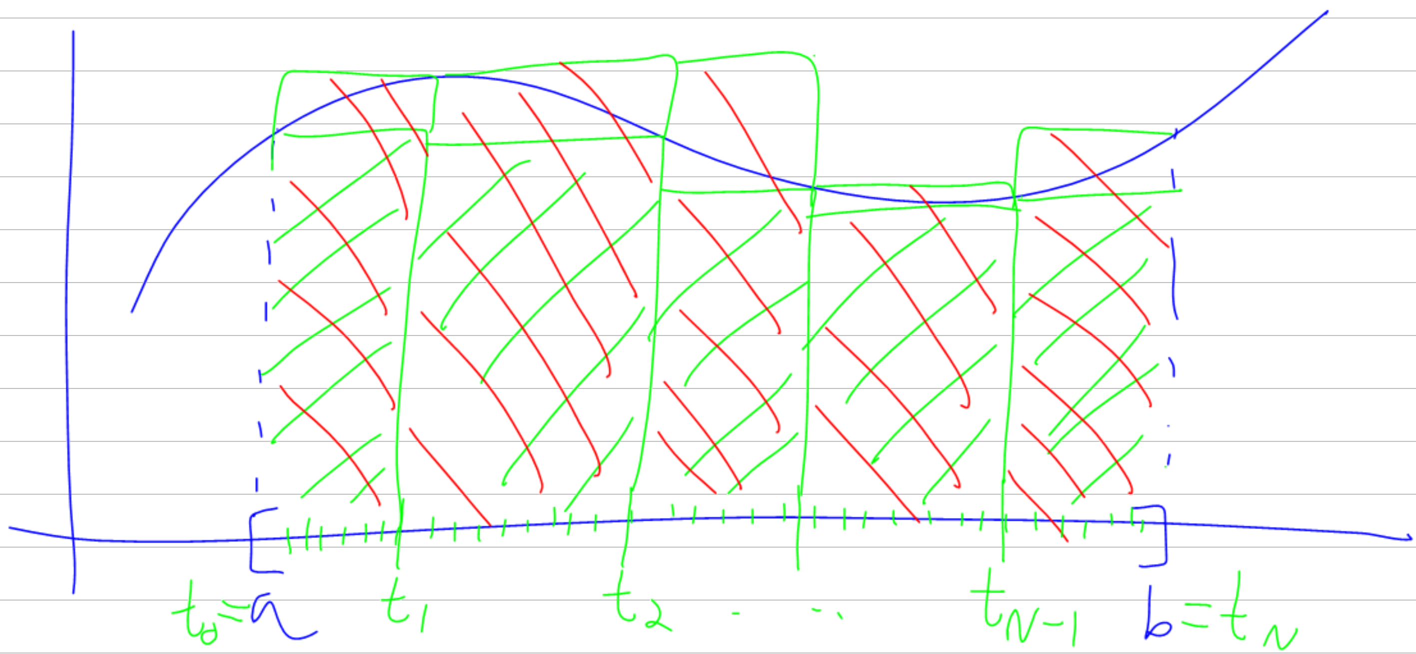
So $$\frac{\partial g}{\partial x} = -\left(\frac{\partial F}{\partial y}\right)^{-1}\frac{\partial F}{\partial x}$$

Stokes' Thm $$\int_M d\omega = \int_{\partial M} \omega$$

infrastructure for d.

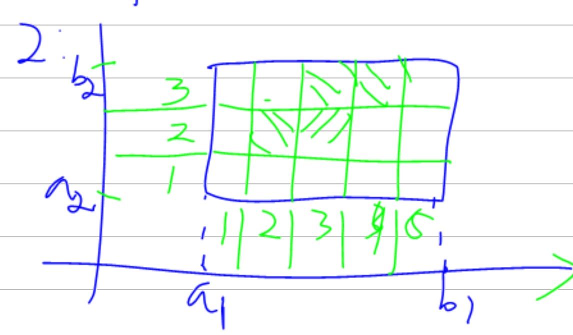
Now: infrastructure for integration $F:\mathbb{R}^n \to \mathbb{R}$

$F:\mathbb{R}\to\mathbb{R}$ really $F:[a,b]\to\mathbb{R}$



Now in $\mathbb{R}^n$: Let $R=\prod_{i=1}^{n}[a_i,b_i]$

Let $F: R \to \mathbb{R}$ be a bndd function.



Goal: $\int_R F$

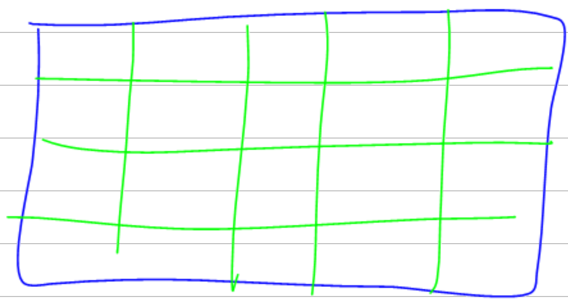
A partition of R is a list $(P_i)_{i=1}^{n}$ where each P_i is a partition of $[a_i, b_i]$:

$$a_i = t_{i,0} < t_{i,1} < \cdots < t_{i,N_i-1} < t_{i,N_i} = b_i$$

disjoint except bndries

R is now the nearly-disjoint union of subrectangles defined by P. Given $1 \leq j_i \leq N_i$, for each $1 \leq i \leq n$, the corresp. subrectangle is

$$S = S_j = \prod_{i=1}^{n} [t_{i,j_i-1}, t_{i,j_i}]$$



<u>Def</u> $$V(R) = \prod_{i=1}^{n} (b_i - a_i) \in \mathbb{R}_{>0}$$

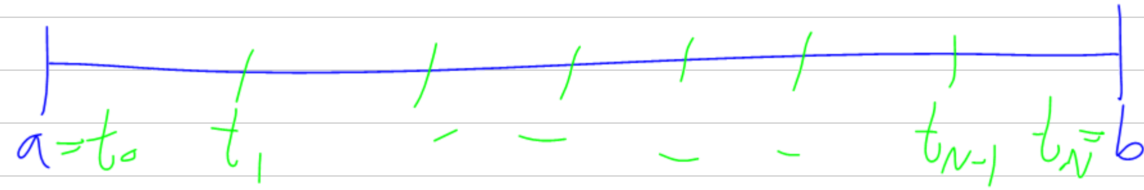
So $$V(S_j) = \prod_{i=1}^{n} (t_{i,j_i} - t_{i,j_i-1})$$

<u>Claim</u> Given R & a partition P thereof,

$$V(R) = \sum_{S \in P} V(S)$$

"S is a subrectangle of R given by the partition P"

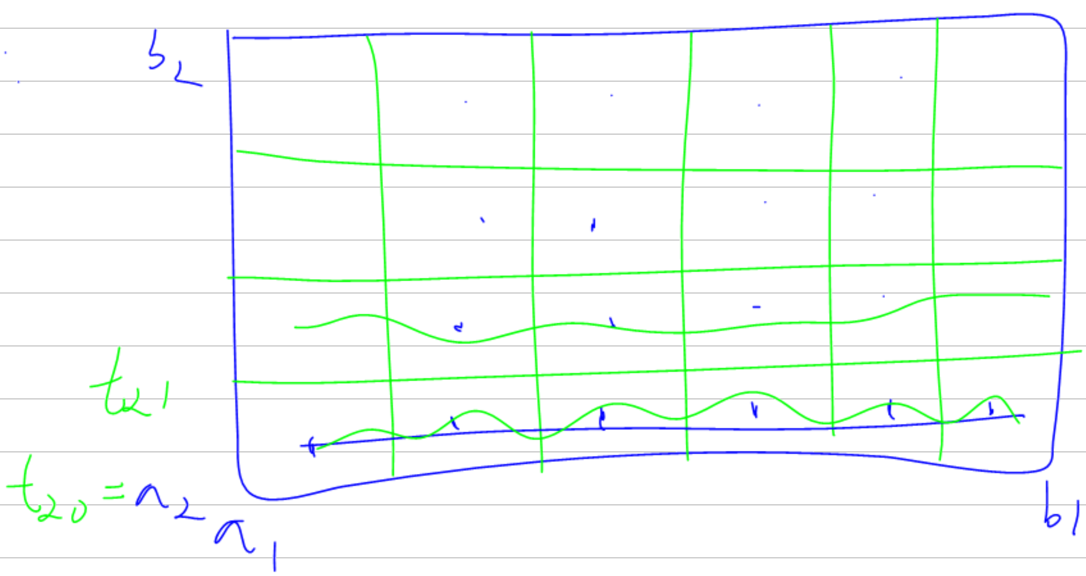
PF For $n=1,2$



$$V(R) \overset{?}{=} \sum_S V(S)$$

$$V(R) = \prod_{i=1} (b_i - a_i) = b - a \overset{?}{=} \sum_j (t_j - t_{j-1}) = t_N - t_0 = b - a \checkmark$$

2:



$(t_{2,2}-t_{2,1})(b_1-a_1)$

$(t_{2,1}-t_{2,0})$

$$(b_1-a_1)(b_2-a_2) = \frac{(b_1-a_1)}{}$$

$$= (b_2-a_2)(b_1-a_1)$$