Term Test 3

Due: Tuesday March 9, 2021 7:20 PM (Eastern Standard Time)

Assignment description

Solve all 5 problems on this test, and do Task 6.

Each problem is worth 20 points.

You have two hours to write this test, and another 20 minutes for Task 6 and for uploading.

Allowed material. Open book(s) and open notes, but you can only use the internet (during the test) to read the test, to submit your solutions, and to connect with the instructor to ask clarification questions. No contact is allowed with other students or with any external advisors, online or in person.

Neatness counts! Language counts! The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and consisting of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (20 points)

Suppose V and W are finite dimensional vector spaces with bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ respectively and dual bases $(\phi_1, \ldots, \phi_n)$ and $(\psi_1, \ldots, \psi_m)$ respectively. Let B be the vector space of bilinear maps $V \times W \to \mathbb{R}$. Suggest a basis for B, prove your suggestion, and determine the dimension of B.

Tip. Don't start working! Read the whole test first. You may wish to start with the questions that are easiest for you.

Q2 (20 points)

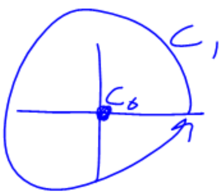
Let $\omega = xy\,dx + 2x\,dy - y\,dz \in \Omega^1(\mathbb{R}^3_{x,y,z})$ and let $\alpha\colon \mathbb{R}^2_{u,v} \to \mathbb{R}^3_{x,y,z}$ be given by $\alpha(u,v) = (uv, u^2, 3u+v)$. Compute $d\omega$, $\alpha^*\omega$, $d(\alpha^*\omega)$, and $\alpha^*(d\omega)$.

Tip. No instructions were given as for *how* to compute, so you can use any method you wish. Though you'd probably be wise to sketch your work: without that, you forfeit any chance of getting partial credit in case your end results are wrong.

Q3 (20 points)

Explain in detail how the vector-field operator curl arises as an instance of the exterior derivative operator $d\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$, for some k and n.

Reminder. $\operatorname{curl}(F_1,\ F_2,\ F_3) = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3},\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1},\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)$.

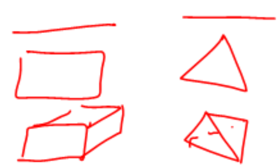


Q4 (20 points)

Let $c_0, c_1 \in C_1(\mathbb{R}^2)$ be given for $t \in [0,1]$ by $c_0(t) = (0,0)$ (a constant!) and by $c_1(t) = (\cos 2\pi t, \sin 2\pi t)$. Show that there is some $c \in C_2(\mathbb{R}^2)$ such that $\partial c = c_1 - c_0$.

Hint. Think polar. $\partial c_1 = 0$ $\quad 1 \cdot c_1 - 1 \cdot c_0$

Note. At the start of the test there was a typo and c_1 was defined without the 2π factors. I apologize.



$c(r,t) = (r\cos 2\pi t, r\sin 2\pi t)$

Q5 (20 points)

Given a smooth cube $c\colon I^k \to \mathbb{R}^n$, a k-form $\omega \in \mathbb{R}^n$, and a smooth bijection $r\colon I^k \to I^k$ for which $\forall p \in I^k\ \det(r'(p)) > 0$, show that $\int_c \omega = \int_{c\circ r} \omega$.

Tip. Once you have finished writing a test, if you have time left, it is always a good idea to go back and re-read and improve everything you have written, and perhaps even completely rewrite any parts that came out messy.

Task 6 (0 points)

Please copy in your own handwriting, fill in the missing details, and sign the statement below, and then submit it along with a photo of your student ID card to complete this test. (See a sample submission below)

By signing this statement, I am attesting to the fact that I, [name], [student number], have abided fully to the Code of Behaviour on Academic Matters. I have not committed academic misconduct, and I am aware of the penalties that may be imposed if I have committed an academic offence.

Signature:

By signing this statement, I am attesting to the fact that I, Dror Bar-Natan, 123456789, have abided fully to the Code of Behaviour on Academic Matters. I have not committed academic misconduct, and I am aware of the penalties that may be imposed if I have committed an academic offence.

Dror Bar-Natan