Final Assessment

Due: Tuesday April 20, 2021 12:20 PM (Eastern Daylight Time)

Assignment description

Solve all 8 problems on this assessment, and do Task 9.

Each problem is worth 20 points, even though the problems are of unequal difficulty.

You have three hours to write this assessment, and another 20 minutes for Task 9 and for uploading.

Allowed material. Open book(s) and open notes, but you can only use the internet (during the assessment) to read the assessment, to submit your solutions, and to connect with the instructor to ask clarification questions. No contact is allowed with other students or with any external advisors, online or in person.

Neatness counts! Language counts! The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and consisting of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (20 points)

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable at a point $a \in \mathbb{R}^n$, and for $b \in \mathbb{R}^n$ define $L(b) := \lim_{\epsilon\to 0} \frac{f(a+\epsilon b)-f(a)}{\epsilon}$. Prove that L is a linear function of b.

Tip. Don't start working! Read the whole assessment first. You may wish to start with the questions that are easiest for you.

Q2 (20 points)

A function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies $|(x-y)-(f(x)-f(y))| \le \frac{1}{3}|x-y|$ for every $x, y \in \mathbb{R}^n$. Prove that f is continuous.

Q3 (20 points)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function whose support is contained in the unit square $[0,1]\times[0,1]$ in $\mathbb{R}^2$, let $s\colon [0,1] \to [0,1]$ be a continuous function, and define $g\colon \mathbb{R}^2 \to \mathbb{R}$ by $g(x,y) = f(x, y-s(x))$. Explain why f is integrable on $[0,1]\times[0,1]$ and why g is integrable on $[0,1]\times[0,2]$, and show that

$$\int_{[0,1]\times[0,1]} f = \int_{[0,1]\times[0,2]} g.$$

Hint. Make Fubini happy!

Q4 (20 points)

If $\gamma: \mathbb{R} \to \mathbb{R}^n$ is a smooth path and $t \in \mathbb{R}$, let $\dot{\gamma}(t)$ be the tangent vector to $\mathbb{R}^n$ given as the pair $(\gamma(t), \gamma'(t)e_1)$, where e_1 is the standard basis vector of $\mathbb{R}$. Show

(a) If $f: \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then $D_{\dot{\gamma}(t)} f = (f \circ \gamma)'(t)$, where D denotes the directional derivative.

(b) If $g: \mathbb{R}^n \to \mathbb{R}^m$ is smooth and $\eta = g \circ \gamma$, then $\dot{\eta}(t) = g_*(\dot{\gamma}(t))$.

Q5 (20 points)

Let $v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. Together, they form a basis (v_1, v_2) of $\mathbb{R}^2$. Write the dual basis (φ_1, φ_2) as a pair of row vectors.

Comment. Until 10AM at exam time the question had "raw" instead of "row", a silly typo.

Q6 (20 points)

Let c_1 and c_2 be singular 1-cubes in $\mathbb{R}^7$, for which $c_1(0) = c_1(1)$ and $c_2(0) = c_2(1)$.

(a) Show that there is a singular 2-cube c in $\mathbb{R}^7$ for which $\partial c = c_1 - c_2$.

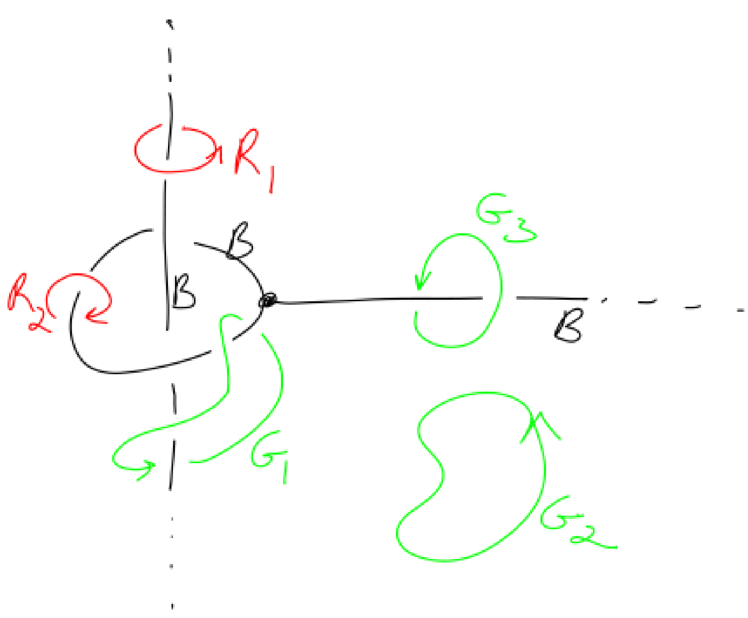
(b) Suppose now that $c_1(0) = c_1(1)$ but $c_2(0) \neq c_2(1)$. Is it still possible that there is a singular 2-cube c in $\mathbb{R}^7$ for which $\partial c = c_1 - c_2$?

Q7 (20 points)

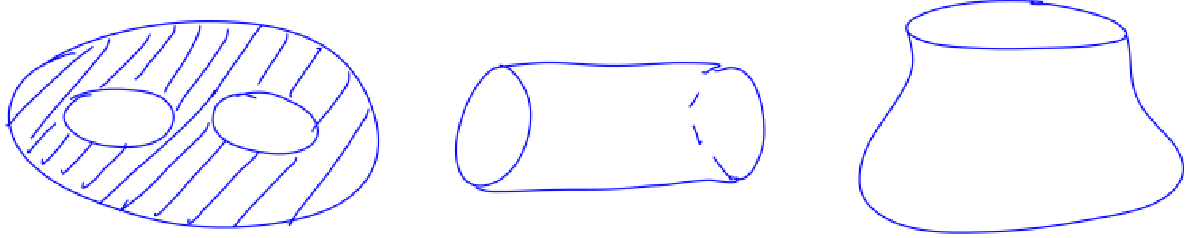
Suppose a k-dimensional manifold M in $\mathbb{R}^n$ is given near a point $p \in M$ as the zero set of a function $z: \mathbb{R}^n \to \mathbb{R}^{n-k}$ whose differential is of maximal rank at p, and let $\xi \in T_p\mathbb{R}^n$. Show that $\xi \in T_pM$ if and only if $z_*\xi = 0$. (In doing so, you will have to recall the definition of T_pM!)

Q8 (20 points)

A subset B of $\mathbb{R}^3$ is the union of an infinite line, an infinite ray, and a circle positioned as on the figure below. In addition, oriented loops R_1, R_2, G_1, G_2, and G_3 are also given as in the same figure. A closed $\omega \in \Omega^1(\mathbb{R}^3 \setminus B)$ is also given, and it is known that $\int_{R_1} \omega = \pi$ and $\int_{R_2} \omega = e$. Compute $\int_{G_i} \omega$ for $i = 1, 2, 3$.



Hint. You may want to also think about 2D subsets of $\mathbb{R}^3$ that are shaped like masks, tubes, and/or sacks as in the figure below.



Tip. Once you have finished writing an assessment, if you have time left, it is always a good idea to go back and re-read and improve everything you have written, and perhaps even completely rewrite any parts that came out messy.

Task 9 (0 points)

Please copy in your own handwriting, fill in the missing details, and sign the statement below, and then submit it along with a photo of your student ID card to complete this test. (See a sample submission below)

By signing this statement, I am attesting to the fact that I, [name], [student number], have abided fully to the Code of Behaviour on Academic Matters. I have not committed academic misconduct, and I am aware of the penalties that may be imposed if I have committed an academic offence.

Signature: