

Read Along: Spivak 66-74.

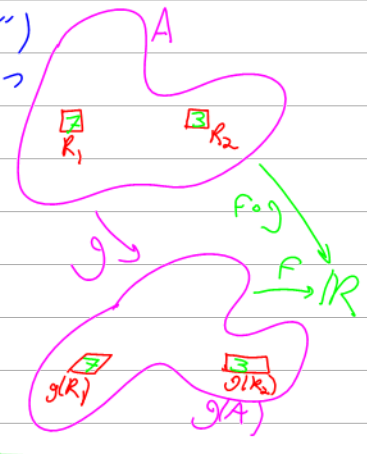
TT2 next week, discussion Wednesday.

COV Strategy: Show that every g is a composition of layer-preserving maps, and use dimensional reduction on those.

Where we were:

Thm (Change of Variables, "COV")

Let $A \subset \mathbb{R}^n$ be open, $g: A \rightarrow \mathbb{R}^n$ cont. diffable, 1-1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $F: g(A) \rightarrow \mathbb{R}$ is integrable, then $\int_{g(A)} F = \int_A (F \circ g) |\det g'|$



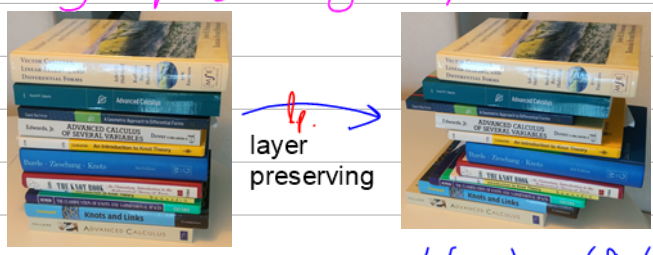
Debt's.

- ~~1. COV(N-1) \Rightarrow COV(N) for l.p. maps~~
- ~~2. Every g is a composition of l.p. maps & coord swaps~~

- ~~3. COV(g), COV(h) \Rightarrow COV(goh)~~
- 4. COV holds for coordinate swaps
- 5. local COV \Rightarrow global COV

- ~~6. Prove COV(1)!~~
- 7. COV(cont) \Rightarrow COV(integ)

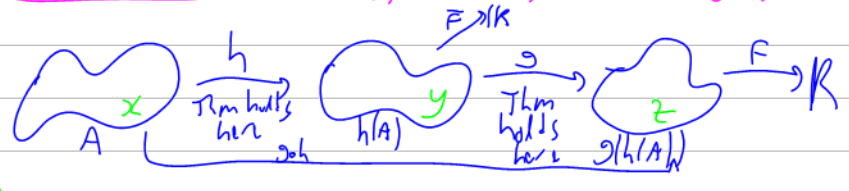
Layer preserving maps:



$$g: \mathbb{R}_x^n \rightarrow \mathbb{R}_y^n \quad g \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} g_1(x) \\ \vdots \\ g_{n-1}(x) \\ x_n \end{pmatrix}$$

$$g(x) = x_n$$

Lemma 3 $COV(g), COV(h) \Rightarrow COV(goh)$



$$\begin{aligned} \text{PF} \int_{g(h(A))} F &\stackrel{COV(g)}{=} \int_{h(A)} \underbrace{(F \circ g)}_F |\det g'| &\stackrel{COV(h)}{=} \int_A (F \circ h) |\det h'| \\ &= \int_A (F \circ goh) |\det(g' \circ h)| |\det h'| \\ &= \int_A (F \circ goh) \cdot |\det(g' \circ h) \cdot h'| = \int_A (F \circ goh) \cdot |\det(goh)'| \quad \square \end{aligned}$$

COV(N-1) \Rightarrow COV(N)

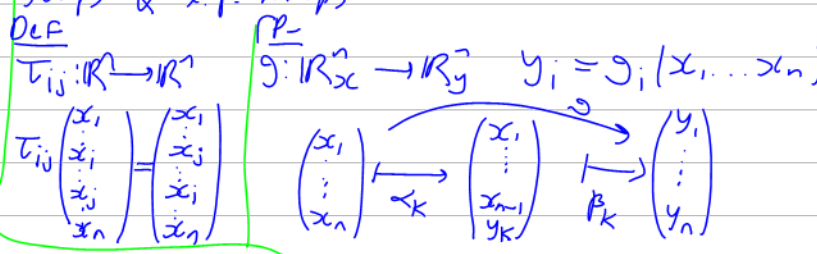
Lemma 1 Assume COV(N-1) Let $U \subset \mathbb{R}^n$ be open & bndd. Let $g: U \rightarrow \mathbb{R}^n$ be a l.p. map cont. diffable, 1-1, $g'(x)$ invertible & $g(U)$ is bndd. Then a restricted COV holds. Namely, if $F: g(U) \rightarrow \mathbb{R}$ is bndd & cont. & $\text{supp } F \subset g(U)$, then $\int F = \int (F \circ g) |\det g'|$

Lemma 2 In the conditions of the main thm, For every $a \in A$ there is some open $U \ni a$, s.t. on U g is a composition of coord. swaps & l.p. maps.

Proof Use Fubini on both sides.

Lemma 6. COV(1).

PF. Follows from the Fundamental thm of Calculus



Lemma 5 Local cov \Rightarrow global cov:

Find a cover $\mathcal{V} = \{V\}$ of $g(A)$ by bndd open sets s.t. $\forall V \in \mathcal{V}$ $g^{-1}(V)$ is bndd & on it g

is a composition of l.p. maps & coord. swaps.

Let $\{\psi_i\}$ be a POI for $g(A)$ sub to \mathcal{V} . here we use g is a bijection!

Then $\{\psi_i \circ g\}$ is a POI for A sub to $\mathcal{U} = \{g^{-1}(V)\}$

So

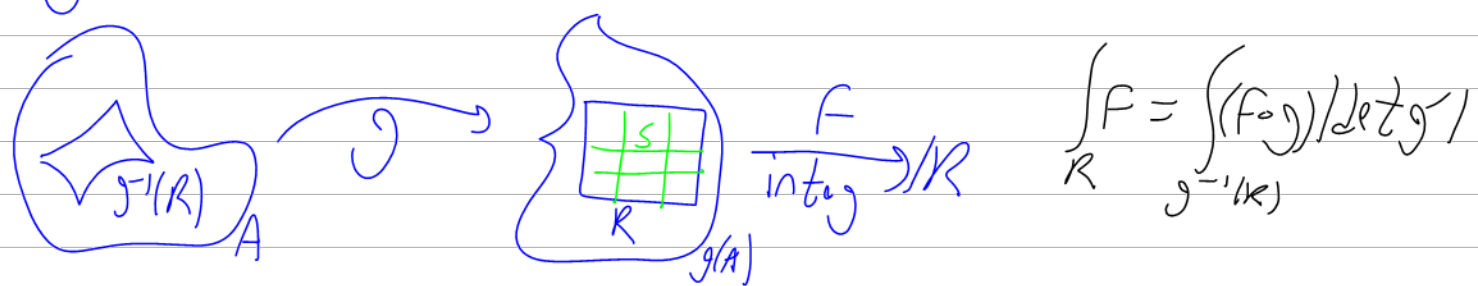
$$\int_{g(A)} F = \sum_i \int_{\mathbb{R}^n} \psi_i F = \sum_i \int_{\mathbb{R}^n} (\psi_i \circ g)(F \circ g) |\det g'| = \dots$$

Lemma 7 Suppose cov holds for ~~cont~~ ^{const} functions f .

Then it also holds for arbitrary integrable f .

PF We'll prove a local version which can be globalized as before

NJS:



PF $L(F, P) = \sum_{SEP} v(S) m_S(F) = \sum_S \int_S m_S(F) =$

$$= \sum_S \int_{g^{-1}(S)} m_S(F) |\det g'| \leq \sum_S \int_S (F \circ g) |\det g'|$$

$$= \int_{g^{-1}(R)} \chi_S(F \circ g) |\det g'|$$

$$\leq \int_{g^{-1}(R)} \sum_S \chi_S(F \circ g) |\det g'|$$

$$= \int_{g^{-1}(R)} (F \circ g) |\det g'| \leq \int_{\dots} V(F, P) \square$$

Aside $f_{h_1} + f_{h_2} \leq f_{(h_1+h_2)}$

PF

$$L(h_1+h_2, P) = \sum_S V(S) m_S(h_1+h_2)$$

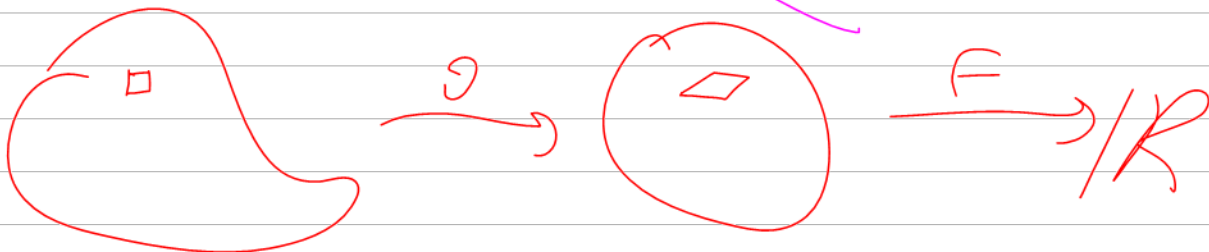
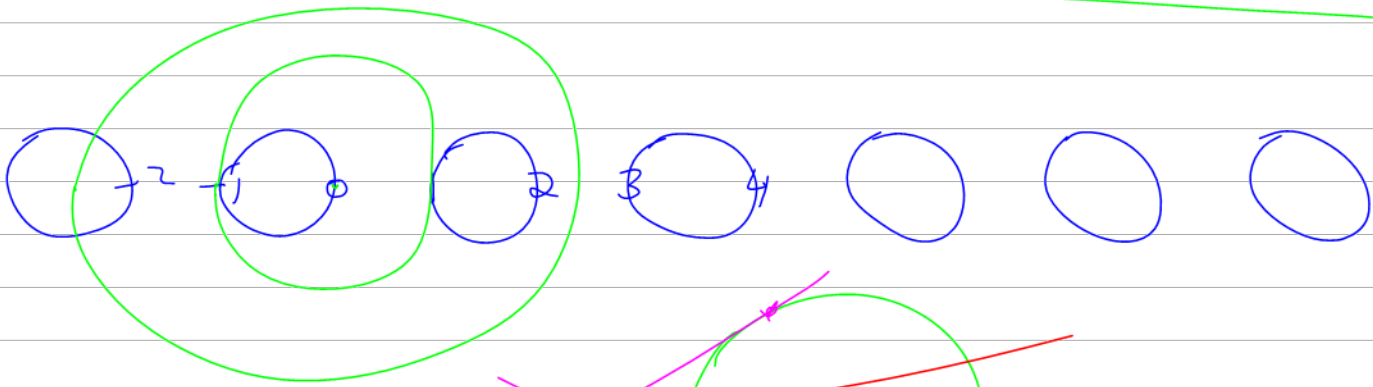
$$\geq \sum_S V(S) (m_S(h_1) + m_S(h_2))$$

$$= L(h_1, P) + L(h_2, P)$$

Lemma 4 cov holds for coord swaps T_{ij} .

PF NTS $\int_{T_{ij}(A)} F = \int_A F \circ T_{ij}$ E.g., $\int_{TA} F(x,y) = \int_A F(y,x)$

COV



Layer-preserving Fnon:

$$g \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \\ y_k \\ * \\ x_n \end{pmatrix}$$



$$g: \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad \begin{matrix} y_1 = g_1(x_1, \dots, x_n) \\ \vdots \\ y_n = g_n(x_1, \dots, x_n) \end{matrix}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_k \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$



Dear All,

Due to clashes of my current office hours schedule with MAT267, as of next week my office hours will take place on Tuesdays at 9-10 (as before) and at 12-1 (replacing 1-2).

Here's what I'll say in class tomorrow about Term Test 2.

- The test will take place on Tuesday January 19, 5-7PM (Toronto time), on Crowdmark (you will get a link by email about one minute before the official starting time). Other than documented accessibility matters, no exceptions!
- Our TAs Sebastian and Shuyang will hold extra pre-test office hours, in their usual zoom rooms. Sebastian on Monday 11-2 at [Sebastian's Zoom](https://us02web.zoom.us/j/88310099689?pwd=S1dTNmJDZkRBNW5QWENWcWtyWm5QQT09) (<https://us02web.zoom.us/j/88310099689?pwd=S1dTNmJDZkRBNW5QWENWcWtyWm5QQT09>) (password vchat), and Shuyang on Friday and on Tuesday at 10:30-11:30 at [Shuyang's Zoom](https://utoronto.zoom.us/j/83428997680) (<https://utoronto.zoom.us/j/83428997680>) (password vchat). These office hours replace some of their regular office hours; so Sebastian will not hold his regular office hours on January 25 and on February 1, and Shuyang will not hold her regular office hours on January 20 and 27.
- I will hold my regular office hours on Tuesday at 9-10 and 12-1, at <http://drorbn.net/vchat> (<http://drorbn.net/vchat>).
- I will be available to answer questions throughout the exam, at my usual office (<http://drorbn.net/vchat> (<http://drorbn.net/vchat>), but I'll add a waiting room). I will also be monitoring my regular email address (drorbn@math.toronto.edu (<mailto:drorbn@math.toronto.edu>)) throughout the exam.
- There will be mishaps! I just hope that not too many. If you encounter one, document everything with specific details, times, and screen shots, and send me a message by Wednesday January 20 at 7PM. I will deal with these situations on a case by case basis.
- Don't let unanswered questions and/or mishaps paralyze you! If you need an answer but for whatever reason you cannot reach me, think hard, come up with what you think is the most reasonable answer/resolution, document as best as you can (for example, by adding a note on your submission), and act following your conclusions.
- Material: Everything up to and including the Change of Variables Theorem, with greater emphasis on the material that was not included in Term Test 1 (meaning, starting with integration).
- Open book(s) and open notes but you can only use the internet (during the exam) to read the exam, to submit the exam, and to connect with the instructor/TAs to ask clarification questions. No contact allowed with other students or with any external advisors, online or in person.
- The format will be "Solve 7 of 7", or maybe "6 of 6" or "5 of 5".
- You will be required to copy in your handwriting and sign an academic integrity statement and submit it on Crowdmark along with the rest of your exam. If you wish, you may save time by preparing the academic integrity statement in advance as in [this sample](http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT1-LastTask.png) (<http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT1-LastTask.png>).
- You will be given an extra 20 minutes at the end of the exam to upload it and to copy/sign the academic integrity statement.
- The vast majority of students will do honest work, and I appreciate that. Out of respect for the honest students I will do my best to pursue and punish any cheating that may occur. I'm more experienced than you! If you plan to be dishonest, think again.
- To prepare: Do the TT2 "rejects" available [here](http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT2-Rejects.pdf) (<http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT2-Rejects.pdf>), but more important: make sure that you understand every single bit of class material so far!
- It is not the test I want! Class material and HW are important, but there won't be questions straight from class/HW. Many things in 2020/21 are not as we want them.

Best,

Dror.

Read Along: Spivak 66-74.

TT2 next week on Tuesday, 5-7PM.

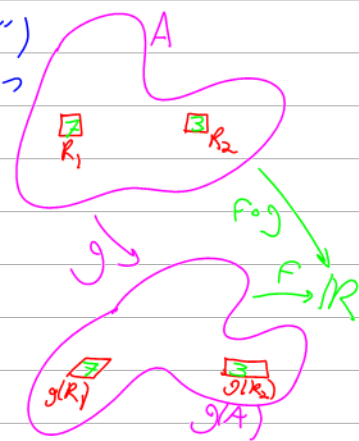
COV Strategy: Show that every g is a composition of layer-preserving maps, and use dimensional reduction on those.

Where we were:

Thm (Change of Variables, "COV")

Let $A \subset \mathbb{R}^n$ be open, $g: A \rightarrow \mathbb{R}^n$ cont. diffable, 1-1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $F: g(A) \rightarrow \mathbb{R}$ is integrable, then

$$\int_{g(A)} F = \int_A (F \circ g) |\det g'|$$



Depts.

~~1. COV(n-1) \Rightarrow COV(n) for 0-p. maps~~

~~2. Every g is a composition of 0-p. maps & coord swaps.~~

~~3. COV(g), COV(h) \Rightarrow COV($g \circ h$).~~

4. COV holds for coordinate swaps

~~5. local COV \Rightarrow global COV.~~

~~6. Prove COV(1)!~~

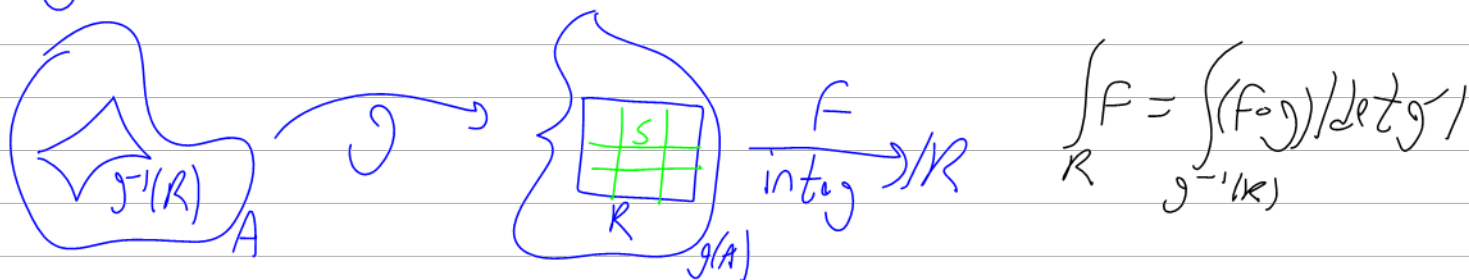
7. COV(cont) \Rightarrow COV(integ)

Lemma 7 Suppose COV holds for ~~cont~~ ^{const} functions F .

Then it also holds for arbitrary integrable F .

PF We'll prove a local version which can be globalized as before

NTS:



PF
$$L(F, P) = \sum_{SEP} v(S) m_S(F) = \sum_S \int_S m_S(F) =$$

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$$= \int_{g^{-1}(R)} (F \circ g) |\det(g')| \leq \int_{\dots} \leq V(F, P) \quad \square$$

Aside $f_{h_1} + f_{h_2} \leq f_{(h_1+h_2)}$

PF

$$L(h_1+h_2, P) = \sum_S V(S) m_S(h_1+h_2)$$

$$\geq \sum_S V(S) (m_S(h_1) + m_S(h_2))$$

$$= L(h_1, P) + L(h_2, P)$$

Lemma 4 cov holds for coord swaps T_{ij} .

PF NTS $\int_{T_{ij}(A)} F = \int_A F \circ T_{ij}$ E.g., $\int_{TA} F(x,y) = \int_A F(y,x)$

COV TT2

The Baby Sard Theorem. $A \subset \mathbb{R}^n$ open, $g: A \rightarrow \mathbb{R}^m$ cont diffable, $C = \{x \in A : \det(g'(x)) = 0\}$.

Then $g(C)$ is of meas-0.

Corollary cov holds even w/o the condition "g'(x) is invertible".

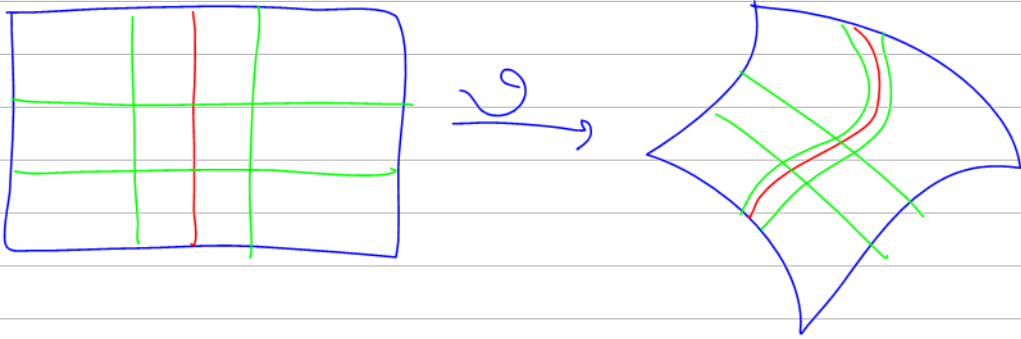
The Adult Sard Thm (Way harder!) $g: A \xrightarrow{\text{open } \mathbb{R}^n} \mathbb{R}^m$

k-times cont. diffable where $k = \max(n-m+1, 1)$,

$C := \{x : \text{rank } g'(x) < m\}$. Then $g(C)$ is meas-0.

Proof of Baby Sard. WLOG A is a closed rectangle.

Pick $\epsilon > 0$, Find a partition P of A s.t. if $S \in P$ & $S \cap C \neq \emptyset$, then all $|D_i g_j| < \epsilon$ on S .



Read Along: Spivak 66-74.

TT2 next week on Tuesday, 5-7PM.

Riddle Along: Four points form a perfect square in the plane. Can you turn that square into a larger one by a sequence of moves of the form $(x,y) \rightarrow (x,2x-y)$ performed on pairs from within our four points? (Namely, by reflections of y about x ?)

4. COV holds for coordinate swaps

Lemma 4 COV holds for coord swaps T_{ij} .

PF NTS $\int_A F = \int_{T_{ij}(A)} F \circ T_{ij}$ EG, $\int_{TA} F \circ T = \int_A F$

Q. How do you write the proof of something so disturbingly obvious?
A. You go back to the defs.

PF Given $A = [a_1, b_1] \times [a_2, b_2]$; $TA = [a_2, b_2] \times [a_1, b_1]$

$P = ((a_1 = t_{10}, t_{11}, \dots, t_{1n} = b_1), (a_2 = t_{20}, \dots, t_{2n} = b_2))$ $TP = \dots$

$L(F \circ T, P) = \sum_{SEP} v(s) m_s(F \circ T) = \sum_{SEP} v(s) m_{Ts}(F) =$

$\sum_{T \in SEP} \dots = \sum_{SEP} \dots = L(F, TP)$

4 COV

The Baby Sard Theorem. $A \subset \mathbb{R}^n$ open, $g: A \rightarrow \mathbb{R}^m$ cont. diffable, $C = \{x \in A : \det(g'(x)) = 0\}$.

Then $g(C)$ is of mens-0.

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The Adult Sard Thm (Way harder!) $g: A \xrightarrow{\text{open } \mathbb{R}^n} \mathbb{R}^m$

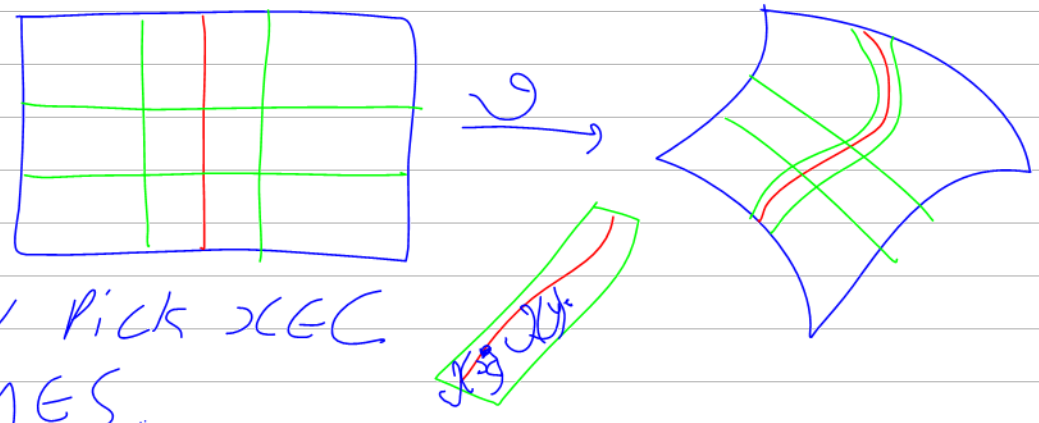
k-times cont. diffable where $k = \max(n-m+1, 1)$,

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Proof of Baby Sard. WLOG A is a closed rectangle.

Pick $\epsilon > 0$, Find a partition P of A s.t. if

$S \in P$ & $S \cap C \neq \emptyset$, then $|g'(x) - g'(y)| < \epsilon$ on S .



Now pick $x \in C$ & $y \in S$,

$$|g(y) - (g(x) + g'(x)(y-x))| < C_1 \epsilon |y-x|$$

$\underbrace{\text{I'd do it, w/ } h(y) = \uparrow, h(x) = 0 \text{ \& } |h'| < \epsilon, \dots}_{\text{lies in an } (n-1)\text{-dim subspace of } \mathbb{R}^n} < C_2 \epsilon d(S)$

$\underbrace{\text{lies in an } (n-1)\text{-dim hypersurface.}}_{\text{lies in an } (n-1)\text{-dim hypersurface.}}$

$\Rightarrow g(S)$ lies in a cylinder of height $\epsilon C_1 d(S)$ and base area $\leq C_2 d(S)^{n-1}$

$\Rightarrow g(S)$ can be covered by rectangles of total volume $\leq C_1 C_2 \epsilon d(S)^n \leq C_3 \text{Vol}(S) \epsilon$

So $g(C)$ can be covered by rects of tot vol

$$\sum_{S: S \cap C \neq \emptyset} C_3 \text{Vol}(S) \epsilon < \sum_{S \in P} C_3 \text{Vol}(S) \epsilon < \text{Vol}(A) C_3 \epsilon \quad \square$$

Please be gentle with zoom chat!

* Class-related only.

* Try not to reach the point where the chat is a distraction.

* Be gentle! Be kind! You don't see everyone - your words may offend somebody you don't even see.

Riddle Along: Can you fold a rectangular piece of paper (perhaps many times) so that the result will have a longer perimeter than the original?

Remember,

$$\int_C dw = \int_C w \frac{\partial C}{\partial C}$$

Def $T: V^k \rightarrow \mathbb{R}$ multi-linear; "k-tensor"

$\mathcal{T}^k(V)$, a v.s.

$$\otimes: \mathcal{T}^k(V) \times \mathcal{T}^l(V) \rightarrow \mathcal{T}^{k+l}(V)$$

multilinear and associative, non-commutative

$$\mathcal{T}^1(V) = V^* \left[\text{Eg. } (\mathbb{R}^2)^* = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}^* = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \right]$$

Thm V w/ basis v_1, \dots, v_n ; $\varphi_1, \dots, \varphi_n$ the dual basis

Then $\{\varphi_{i_1} \otimes \dots \otimes \varphi_{i_k} : 1 \leq i_j \leq n\}$ is a basis of $\mathcal{T}^k(V)$. Hence $\dim \mathcal{T}^k(V) = n^k$

PE ...

Def Given $F: V \rightarrow W$, the pullback $F^*: \mathcal{T}^k(W) \rightarrow \mathcal{T}^k(V)$ (linear, respects \otimes)

Example T is an "inner-product"
Gram-Schmidt

$$F^*T = \langle \rangle$$

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- * Be gentle! Be kind! You don't see everyone - your words may offend somebody you don't even see.

Riddle Along: Let f be a distance-non-increasing function from the plane to the plane ($d(x,y) \geq d(f(x),f(y))$), for all x,y , and let R be a rectangle in the plane. Is it always true that the length of the boundary of R is greater or equal to the length of the boundary of $f(R)$?

Remember,

$$\int_C dw = \int \frac{\partial w}{\partial c}$$

Warning: Elsewhere $\mathcal{T}^k(V^*)$

Recall $\mathcal{T}^k(V) =$ "k-tensors on V " = $\left\{ \begin{array}{l} \text{k-linear maps} \\ V^k \rightarrow \mathbb{R} \end{array} \right\}$
 \sim v.s.!

Also, $\mathcal{T}^k \times \mathcal{T}^l \rightarrow \mathcal{T}^{k+l}$ via $(T_1, T_2) \mapsto T_1 \otimes T_2$

$$\mathcal{T}^1(V) = V^* \left[\text{Eg } (\mathbb{R}^2)^* \cong \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right]^* = \left[\begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix} \right] \right]$$

Then V w/ basis v_1, \dots, v_n ; $\varphi_1, \dots, \varphi_n$ the dual basis

Then $\{\varphi_{i_1} \otimes \dots \otimes \varphi_{i_k} : 1 \leq i_j \leq n\}$ is a basis of $\mathcal{T}^k(V)$. Hence $\dim \mathcal{T}^k(V) = n^k$.

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Example T is an "inner-product"
 Gram-Schmidt, $F^*T = \langle \rangle$

Claim Alternating \Leftrightarrow kills repetitions

Def $\Lambda^k(V) \sim$ sub-v.s. of $\mathcal{T}^k(V)$
 Warning: $\mathcal{T}^k(V^*)$

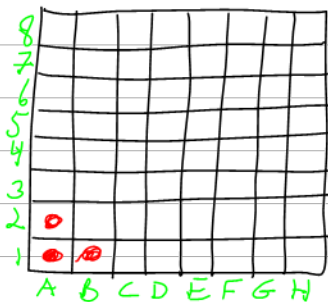
Examples: Determinants, minors.

Permutations & signs.

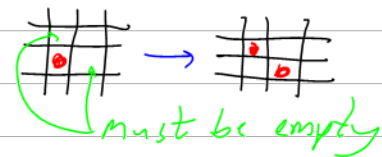
HW11, due next Wednesday, is on the web!

Read Along: Spivak 75-85.

Riddle Along: On a chessboard, there are three pawns at the lower left (at A1, A2, and B1). On each move, pick up one pawn, remove it and place one new pawn to the right and one new pawn above, but only if these squares are unoccupied. Can you clear the original 3 pawns?



A move:



Notation $\underline{n} = \{1, \dots, n\}$ $\underline{I} = I \in \underline{n}^k$ means $I = \underline{J} = (i_1, \dots, i_k)$

If $v_j \in V$, $V_{\underline{I}} = (v_{i_1}, \dots, v_{i_k}) \in V^k$

If $\varphi_j \in V^*$, $\varphi_{\underline{I}} = \varphi_{i_1} \otimes \dots \otimes \varphi_{i_k} \in \mathcal{T}^k(V)$

Thm V w/ basis v_1, \dots, v_n ; $\varphi_1, \dots, \varphi_n$ the dual basis

Then $\{\varphi_{\underline{I}} : \underline{I} \in \underline{n}^k\}$ is a basis of $\mathcal{T}^k(V)$.

Hence $\dim \mathcal{T}^k(V) = n^k$.

Done: 1. $T_1 = T_2$ in $\mathcal{T}^k(V)$ iff $\forall \underline{I} T_1(\underline{I}) = T_2(\underline{I})$

2. $\varphi_{\underline{I}}(v_{\underline{J}}) = \delta_{\underline{I}\underline{J}}$.

span Given $T \in \mathcal{T}^k(V)$, we want $T = \sum_{\underline{I} \in \underline{n}^k} a_{\underline{I}} \varphi_{\underline{I}}$

Then L.I.

Def Given $F: V \rightarrow W$, the pullback

(linear, respects \otimes)

$$F^*: \mathcal{T}^k(W) \rightarrow \mathcal{T}^k(V)$$

Example T is an "inner-product"

Gram-Schmidt, $F^*T = \langle \rangle$

Claim Alternating \Leftrightarrow kills repetitions

Def $\Lambda^k(V) \subset \mathcal{T}^k(V)$ warning: $\mathcal{T}^k(V^*)$ sub-v.s. of $\mathcal{T}^k(V)$

Examples: Determinants, minors.

Permutations & signs.

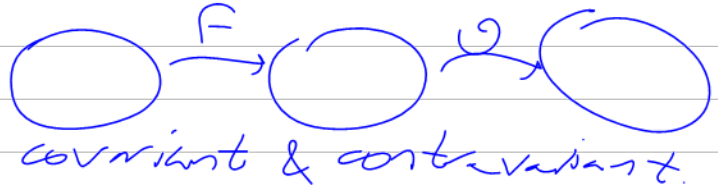
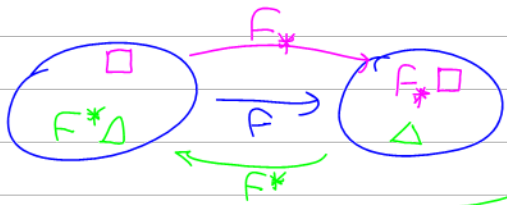
TT2: Not yet.

Read Along: Spivak 75-85.

Riddle Along: The Cantor set C is of measure 0. Is the same true for $C+C = \{x+y : x, y \in C\}$?

Recall: $I, J \in \underline{n}^k$, $\varphi_I = \varphi_{i_1} \otimes \dots \otimes \varphi_{i_k}$, $V_I = (v_{i_1}, \dots, v_{i_k})$, $\varphi_I(V_J) = \delta_{IJ}$

$\Rightarrow \varphi_I$ is a basis of $T^k(V)$; $\dim T^k(V) = n^k$



Claim Alternating \Leftrightarrow kills repetitions

Def $\Lambda^k(V) \sim$ sub-v.s. of $T^k(V)$ Warning: $\Lambda^k(V) \neq \Lambda^k(V^*)$

Examples: Determinants, minors.

Permutations [make a group S_k] signs [Axiomatics, standard diagrams, TT side (or side)]

$$T \in \Lambda^k \Leftrightarrow T \circ \sigma = (-1)^{\text{sgn}(\sigma)} T$$

$$(\text{Alt } T)(v_1, \dots, v_k) = \frac{1}{k} \sum_{\sigma \in S_k} \frac{1}{|K|} T(v_{\sigma})$$

Prop 1 $\text{Im Alt} \subset \Lambda^k$, so $\text{Alt}: T^k \rightarrow \Lambda^k$

2. IF $W \in \Lambda^k$, $\text{Alt}(W) = W$

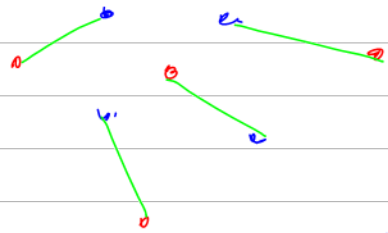
3. $\text{Alt} \circ \text{Alt} = \text{Alt}$

$$W \wedge \eta = \frac{(k+l)!}{k!l!} \text{Alt}(W \otimes \eta)$$

Bilinear, associative, super-commutative.

"commutes" with pullbacks.

Riddle Along: n red points and n blue points are placed in the plane with no 3 on the same line. Prove that it is possible to pair them up using n straight line segments so that no two of the segments will intersect.



Def $\Lambda^k(V) = \{T \in \mathcal{L}^k(V) : T \text{ is alternating / anti-symmetric}\}$

A subspace!

Examples: Determinants & minors!

$$S_k = \{\text{bijections } \underline{k} \rightarrow \underline{k}\}$$

"The k -th symmetric/permutation group"

Thm $\exists \text{! sign} : S_k \rightarrow \{\pm 1\}$ s.t.

$$(-1)^{\sigma \circ \lambda} = (-1)^\sigma (-1)^\lambda, \quad (-1)^{\tau_{ij}} = -1$$

board line

Notes strand diagrams, $\prod_{i < j} \dots$, Det

$$T \in \Lambda^k \Leftrightarrow T \circ \sigma = (-1)^\sigma T$$

$$\underline{\Lambda^k} \sim \binom{n}{k} \subset \underline{\Lambda^k}$$

W_I

basis for $\Lambda^k(V)$

$\dim \Lambda^k(V)$

by imitating $\mathcal{L}^k(V)$

!

IF time, count non-Int seqs.

$$(\text{Alt } T)(v_1 \dots v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn } \sigma T(v_{\sigma(1)} \dots v_{\sigma(k)})$$

Prop 1. $\text{Im Alt} \subset \Lambda^k$, so $\text{Alt} : \mathcal{L}^k \rightarrow \Lambda^k$

2. If $W \in \Lambda^k$, $\text{Alt}(W) = W$

3. $\text{Alt} \circ \text{Alt} = \text{Alt}$

$$W \wedge \eta = \frac{(k+l)!}{k!l!} \text{Alt}(W \otimes \eta)$$

Bilinear, associative, super-commutative.

"commutes" with pullbacks.



Thm $\exists!$ sign: $S_K \rightarrow \{\pm 1\}$ s.t. $(-1)^{\sigma \circ \lambda} = (-1)^\sigma (-1)^\lambda$, $(-1)^{\tau_{ij}} = -1$

Def $\Lambda^k(V) = \{T \in \mathcal{O}^k(V) : \forall \sigma \in S_K, T \circ \sigma = (-1)^\sigma T\}$

$\{1 \leq i_1 < i_2 < \dots < i_k \leq n\} =: \underline{n}^k \sim \binom{n}{k} \subset \mathcal{O}^k$

$\lambda \in \Lambda^k(V)$ is determined by $\lambda(V_I)$, $I \in \underline{n}^k$!

$$W_I := \sum_{\sigma \in S_K} (-1)^\sigma \varphi_I \circ \sigma$$

Claim $W_I \in \Lambda^k(V)$, $W_I(V_J) = \delta_{IJ}$

hence $\{W_I\}_{I \in \underline{n}^k}$ is a basis of $\Lambda^k(V)$, hence $\dim \Lambda^k(V) = \binom{n}{k}$

Aside: $\sigma^*: V^k \rightarrow V^k$ by $(\sigma^* u)_i = u_{\sigma(i)}$ then $(\sigma \circ \tau)^* = \tau^* \circ \sigma^*$

PF of claim

Aside $|\underline{n}^k| = \binom{n+k-1}{k}$ PF $1223555 \leftrightarrow 1*2**3*45***$

Thm $\exists!$ Bilinear $\wedge: \Lambda^k(V) \times \Lambda^l(V) \rightarrow \Lambda^{k+l}(V)$ s.t.

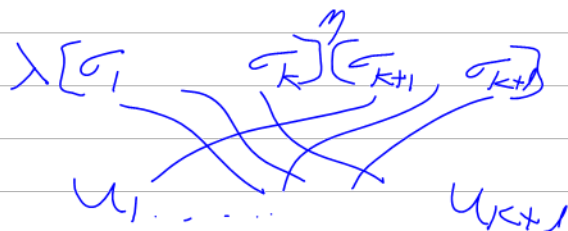
1. Associative

2. Supercommutative

3. $W_I = W_{i_1} \wedge \dots \wedge W_{i_k}$

PF Set $(\lambda \wedge \eta)(u_1 \dots u_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma (\lambda \otimes \eta)(\sigma^*(u_1 \dots u_{k+l}))$

$$= \sum_{\substack{\sigma \in S_{k+l} \\ \sigma(1) < \dots < \sigma(k) \\ \sigma(k+1) < \dots < \sigma(k+l)}} (-1)^\sigma \lambda(u_{\sigma(1)} \dots u_{\sigma(k)}) \eta(u_{\sigma(k+1)} \dots u_{\sigma(k+l)})$$



Hour 46 MAT257 Analysis II on February 1, 2021: More on alternating Tensors: wedge products, minors, pullbacks, degrees 0 and 1, degree n and orientations, degree n-1 and cross products.

Mon -> Mon HW schedule: maybe later. Zoom recording on Friday failed :(.

Read Along: Spivak 75-85.

Riddle Along: Can you colour the points of the plane with 3 colours, so that no two points of distance exactly 1 have the same colour? How about 4, 5, 6, or 7?

Riddled before: Is there a continuous surjection $f: [0, 1] \rightarrow [0, 1]$ which is constant on a set of intervals whose lengths sum to 1?

$$\int_C dw = \int_C w$$

Thm $\exists!$ Bilinear $\wedge: \Lambda^k(V) \times \Lambda^l(V) \rightarrow \Lambda^{k+l}(V)$ s.t.

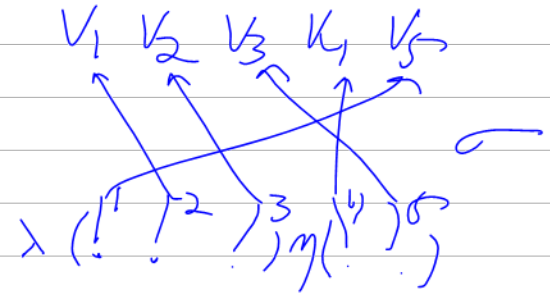
1. Associative.

2. Supercommutative.

3. $w_I = \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$

PF set $(\lambda \wedge \eta)(u_1, \dots, u_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma (\lambda \otimes \eta)(\sigma^*(u_1, \dots, u_{k+l}))$

$$= \sum_{\substack{\sigma \in S_{k+l} \\ \sigma(1) < \dots < \sigma(k) \\ \sigma(k+1) < \dots < \sigma(k+l)}} (-1)^\sigma \lambda(u_{\sigma(1)}, \dots, u_{\sigma(k)}) \eta(u_{\sigma(k+1)}, \dots, u_{\sigma(k+l)})$$



Comments 1 on \mathbb{R}^n , $w_I =$

..... = λ_I of Jun 27.

2. Pullbacks a work.

b. compatible w/ $+$, \cdot , \wedge

c. $L: V \rightarrow V$ $L^* w_{top} = \dots$

may skip { d. $L: V_n \rightarrow V'_n$ $L^* w_{I \in \binom{[n]'}{k}} = \sum_{J \in \binom{[n]'}{k}} C_{IJ} w_J$
 \uparrow
 $\det(A_{IJ})$

3. Λ^0, Λ^1

4. Λ^n and orientations

5. Λ^{n-1} and cross products

$$W_I = \sum_{\sigma} (-1)^{\sigma} (\varphi_{i_1} \wedge \varphi_{i_2} \wedge \dots \wedge \varphi_{i_k}) \circ \sigma^*$$

$$I = (1 \leq i_1 < i_2 < \dots < i_k \leq n)$$

$$\Lambda^k \times \Lambda^l \rightarrow \Lambda^{k+l}$$

1. Assoc

2. Super-commutative

$$3 \quad W_I = \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$$

$$(\wedge^k \eta) \wedge (\wedge^l \eta)(v_1, \dots, v_{k+l})$$

$$= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^{\sigma} \lambda(v_{\sigma(1)} \dots v_{\sigma(k)}) \eta(v_{\sigma(k+1)} \dots v_{\sigma(k+l)})$$

PLV 13: 1. \times in 3D

2. Orientation pres / rev?

$$3. (\wedge^k)^* = \wedge^{n-k}$$

4.

Wed class:
 pf of 1 dim
 pull backs
 Vol elements
 orientations

Riddled before: Is there a continuous surjection $f: [0,1] \rightarrow [0,1]$ which is constant on a set of intervals whose lengths sum to 1?

Riddled before: A unit cube in \mathbb{R}^3 , the area of its projection on any plane is equal to the length of its projection on a perpendicular line to that plane.

Thm $\exists \downarrow$ $(\lambda^k, \eta^l) \mapsto \lambda^k \wedge \eta^l$ s.t.

1. Assoc. $(\lambda \wedge \eta) \wedge \phi = \lambda \wedge (\eta \wedge \phi)$ 2. Super-commutative: $(\lambda \wedge \eta) = (-1)^{kl} \eta \wedge \lambda$

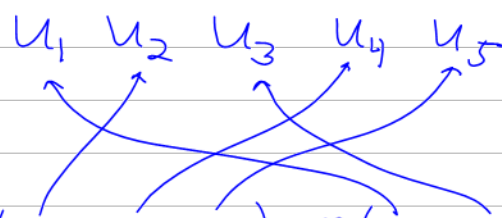
3. $\omega_I = \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$

Existence set $(\lambda \wedge \eta)(u_1, \dots, u_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma (\lambda \circ \eta)(\sigma^*(u_1, \dots, u_{k+l}))$

$= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma \lambda(u_{\sigma(1)}, \dots, u_{\sigma(k)}) \eta(u_{\sigma(k+1)}, \dots, u_{\sigma(k+l)})$

$= \sum_{\substack{\sigma \in S_{k+l} \\ \sigma(1) < \dots < \sigma(k) \\ \sigma(k+1) < \dots < \sigma(k+l)}} (-1)^\sigma \lambda(u_{\sigma(1)}, \dots, u_{\sigma(k)}) \eta(u_{\sigma(k+1)}, \dots, u_{\sigma(k+l)})$

$k=3 \quad l=2:$



on to the proofs of $\in \wedge^{k+l}$ & 1, 2, 3 -

$\sum_{\substack{\text{"good"} \\ \text{"a splitting"}}} (-1)^\sigma \lambda(\dots) \eta(\dots)$ $\sigma = [24513]$

Pullbacks [compatible w/ all ops]

$\dim V = n \quad \wedge^n(V) = \{ \text{Volume elements} \} \cong W$
 $\wedge^{\text{top}}(V)$

$L: V \rightarrow V \Rightarrow L^*W = (\det A)W$

Orientation 1. A basis, up to a positive-det C.O.B.

2. A volume form, up to a pos. scalar

Both pushes & pulls!

Riddled before: Is there a continuous surjection $f: [0,1] \rightarrow [0,1]$ which is constant on a set of intervals whose lengths sum to 1?

Riddled before: A unit cube in \mathbb{R}^3 , the area of its projection on any plane is equal to the length of its projection on a perpendicular line to that plane.

Orientation 1. A volume form, up to a pos. scalar

2. A basis, up to a positive-det C.O.B.

Both pushes & pulls!

$$\mathbb{R}_p^n \sim T_p \mathbb{R}^n = \{(p, v)\} = \{v_i\}$$

a v.s., inner product

Vector fields & component Functns.
(const., differentiable)

$$F(p) = \sum F^i(p) (p, e_i)$$

sums, inner products.

Diff. Forms:

$$\lambda(p) = \sum_{I \in \Omega_n^k} \lambda_I(p) \omega_I(p)$$

Pushing & pulling.

Read Along: Spivak 86-92.

HW14 will be on the web by midnight. It will be due on the next teaching Monday.

Riddled before: Is there a continuous surjection $f: [0, 1] \rightarrow [0, 1]$ which is constant on a set of intervals whose lengths sum to 1?

Riddled before: A unit cube in \mathbb{R}^3 , the area of its projection on any plane is equal to the length of its projection on a perpendicular line to that plane.

Just learned from Tanya Khovanova, <https://blog.tanyakhovanova.com/2021/02/the-anniversary-coin/>: Eight out of sixteen coins are heavier than the rest and weigh 11 grams each. The other eight coins weigh 10 grams each. We do not know which coin is which, but one coin is conspicuously marked as an "Anniversary" coin. Can you figure out whether the Anniversary coin is heavier or lighter using a balance scale at most three times?

Reminder. Given $p \in \mathbb{R}^n$,

$$\mathbb{R}_p^n \sim T_p \mathbb{R}^n := \{(p, v) : v \in \mathbb{R}^n\} = \{v_p\} \quad \text{Pushes!}$$

$$F: \mathbb{R}^n \rightarrow \bigcup_{p \in \mathbb{R}^n} T_p \mathbb{R}^n \quad \text{s.t. } F(p) \in T_p \mathbb{R}^n \text{ is a "vector field"}$$

Can add, scale, inner-multiply, but not push or pull.

$$F(p) = \sum F^i(p) (p, e_i)$$

Direction derivatives $D_{(p,v)}$: "bi"linear, Leibniz.

compatibility w/ push, pull D_F

$$\text{Diff. Forms, } \lambda(p) = \sum_{I \in \Omega_n^k} \lambda_I(p) \omega_I(p)$$

$+$, \cdot , \wedge , pushing & pulling, compatibilities.

$\Omega^k(\mathbb{R}^n)$: cont. diffable k -forms.

$$d: \Omega^0(\mathbb{R}^n) \rightarrow \Omega^1(\mathbb{R}^n)$$

$$dx^i \quad dF$$

Compatibility with pull-backs.

$$\mathbb{R}^n_p \sim T_p \mathbb{R}^n$$

$$= \{(p, v)\} \quad \xi = (p, v)$$

$$(p, v_1) + (p, v_2) = (p, v_1 + v_2)$$

$$D F = F'(p) \cdot v = (F \circ \gamma)'(0)$$

$$\underbrace{(\gamma(0), \dot{\gamma}(0))}_{\xi} = \dot{\gamma}(0)$$

$$F: \mathbb{R}^n \rightarrow \bigcup_p T_p \mathbb{R}^n$$

$$F(p) \in T_p(\mathbb{R}^n) \quad \sim (F')^i$$

$D_F F$ again a function

$$\Omega^k(\mathbb{R}^n) = \int \lambda: \mathbb{R}^n \rightarrow \bigcup_p \Lambda^k(T_p \mathbb{R}^n)$$

$$\lambda = \sum_{I \in \underline{n}^k} \lambda_I \omega_I$$

$$d: \Omega^0 \rightarrow \Omega^1 \quad dx_i$$

$$dF = \sum \frac{\partial F}{\partial x_i} dx_i$$

$$\exists! d: \mathcal{N}^k \rightarrow \mathcal{N}^{k+1}$$

$$d(\omega^{\wedge n}) = - \quad - \quad - \quad .$$

$$d\lambda = \sum dx^i \wedge \frac{\partial \lambda}{\partial x^i}$$

Learned from Tanya Khovanova, <https://blog.tanyakhovanova.com/2021/02/the-anniversary-coin/>: Eight out of sixteen coins are heavier than the rest and weigh 11 grams each. The other eight coins weigh 10 grams each. We do not know which coin is which, but one coin is conspicuously marked as an "Anniversary" coin. Can you figure out whether the Anniversary coin is heavier or lighter using a balance scale at most three times?

Diff. Forms, $\lambda(P) = \sum_{I \in \Omega_n^k} \lambda_I(P) \omega_I(P)$

$+$, \cdot , \wedge , Pushing & pulling, compatibilities.

$\Omega^k(\mathbb{R}^n)$: (smooth) k -Forms.

$d: \Omega^0(\mathbb{R}^n) \rightarrow \Omega^1(\mathbb{R}^n)$ dx^i dF

Compatibility with pull-backs

Example: $\phi: (r, \theta) \mapsto (r \cos \theta, r \sin \theta)$
 $\phi^*(dx dy) = \dots$

Stories about $\int dw = \int w$, then
 Moral definition

$(dw)(\xi_1, \dots, \xi_{k+1}) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{k+1}} \int w$

where $P(\epsilon \xi_1, \dots, \epsilon \xi_{k+1}) = \epsilon \xi_1 \times \dots \times \epsilon \xi_{k+1}$

should be 1. Linear in w 2. Linear in the ξ_i 's
 3. Vanish if repetitions $\Rightarrow dw \in \Omega^{k+1}(\mathbb{R}^n)$

Compute dw up to signs; get

$dw = \sum_{i=1}^n dx^i \wedge \frac{\partial w}{\partial x^i}$ ← The practical def!

Diff. k -Forms on \mathbb{R}^n : $P \mapsto \Lambda^k(T_P \mathbb{R}^n)$ $\Omega^k(\mathbb{R}^n) := \{ \overset{\text{smooth}}{k\text{-forms}} \}$

$d: \Omega^0(\mathbb{R}^n) \rightarrow \Omega^1(\mathbb{R}^n)$ by $df(\xi) = D_\xi f$

dx_i are φ_i so $\lambda \in \Omega^k(\mathbb{R}^n) \Rightarrow \lambda = \sum_{I \in \Omega_n^k} \lambda_I(P) dx_I$

can $+$, \wedge , \lrcorner , pull, all nicely compatible. Compute df !

Example $\phi: (\mathbb{R}_{>0}^2 \rightarrow \mathbb{R}_{x,y}^2$ by $(\theta, r) \mapsto \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$. compute $\phi^*(dx^1 dy)$.

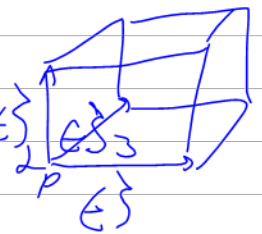
Define $d: \Omega^k \rightarrow \Omega^{k+1}$ by $dw = \sum_{i=1}^n dx_i \wedge \frac{\partial w}{\partial x_i}$

Properties 1. linear 2. Leibnitz 3. pullbacks.

4. $d^2 = 0$

Stories about $\int_C dw = \int_{\partial C} w$, Stokes Theorem: $d = dg$ where

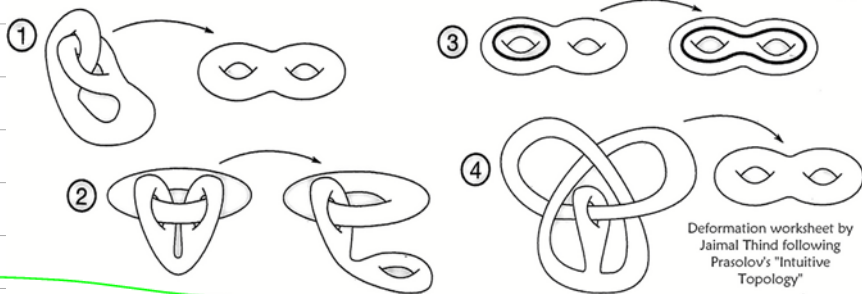
$(dg)(\xi_1, \dots, \xi_{k+1}) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{k+1}} \int w$ $P(\xi_1, \dots, \xi_{k+1}) = \epsilon \xi_1 \wedge \dots \wedge \epsilon \xi_{k+1}$



Should be 1. Linear in w 2. Linear in the ξ_i 's

3. Vanish if repetitions $\Rightarrow dgw \in \Omega^{k+1}(\mathbb{R}^n)$

Compute dgw up to signs!



$$d: \mathcal{N}^k \rightarrow \mathcal{N}^{k+1} \text{ by } dw = \sum_{i=1}^n dx_i \wedge \frac{\partial w}{\partial x_i}$$

Properties \circ At $k=0$ $d_{\text{new}} f = d_{\text{old}} f$. \square

1. Linear $d(w_1 + w_2) = \dots$ $d(\alpha w) = \dots$ \square

2. $d(w \wedge \eta) = dw \wedge \eta + (-1)^{\deg w} w \wedge d\eta$ Leibnitz law \square

3. $d(g^* w) = g^*(dw)$

4. $\mathcal{N}^k \xrightarrow{d} \mathcal{N}^{k+1} \xrightarrow{d} \mathcal{N}^{k+2}$ $d \circ d = 0$
 $d^2 = 0$

Prove 3 & 4, then:

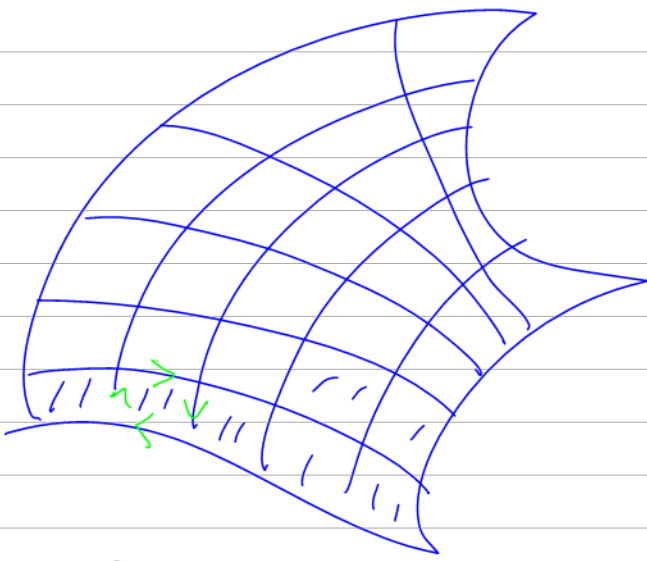
Stories about $\int_C dw = \int_C w$, then Theorem: $d = dg$ where

$$(dg)(\xi_1, \dots, \xi_{k+1}) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{k+1}} \int w$$

$$P(\epsilon \xi_1, \dots, \epsilon \xi_{k+1}) = \epsilon \xi_1 \wedge \dots \wedge \epsilon \xi_{k+1}$$

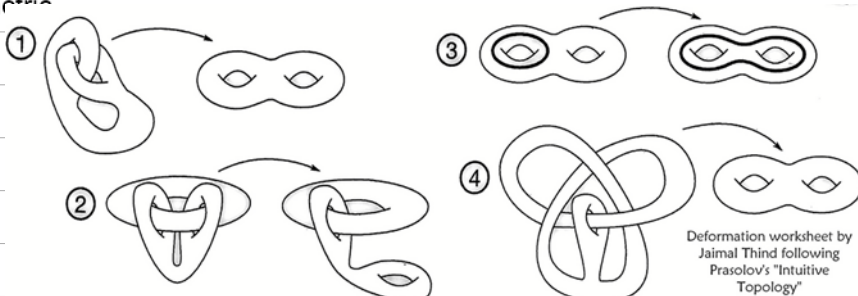
Should be 1. Linear in w 2. Linear in the ξ_i 's
 3. Vanish if repetitions $\Rightarrow dgw \in \mathcal{N}^{k+1}(\mathbb{R}^n)$

Compute dgw up to signs!



$$W = \sum_{i=1}^n (-1)^i \frac{x_i}{|x|^p} dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

For which p is $dW = 0$?



$$d: \mathcal{U}^k \rightarrow \mathcal{U}^{k+1} \text{ by } dw = \sum_{i=1}^n dx_i \wedge \frac{\partial w}{\partial x_i}$$

$df(\xi) = D_\xi f$, linear, (super)-Leibnitz,
 compatible w/ pullbacks, $d^2 = d \circ d = 0$

Thm $(dw)(\xi_1, \dots, \xi_{k+1}) =$ where $\xi_j = (p, v_j)$

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{k+1}} \sum_{i=1}^{k+1} (-1)^{i-1} \left[w(p + \epsilon v_i)(\epsilon v_1, \dots, \epsilon v_i, \dots, v_{k+1}) - w(p)(\epsilon v_1, \dots, \epsilon v_i, \dots, v_{k+1}) \right]$$

(on board ~ sin)

@ $k=0$:

geom meaning, relation w/ $\int_C dw = \int_C w$
Proof Enough to take $w = f \cdot \lambda$, where λ has constant coeffs. Then...

Aside closed & exact, Poincaré's Lemma,

Forms on an open set, $\lambda = \frac{-y dx}{x^2 + y^2} + \frac{x dy}{x^2 + y^2}$

Singular k -cube in $A \subset \mathbb{R}^n$: Cont. $c: [0,1]^k \rightarrow A$
 (0-cubes, 1-cubes)

(The space of k -chains in A) = The free Abelian group generated by all k -cubes = $\left\{ \sum_{i=1}^m a_i c_i \right\}$
 1. order immaterial
 2. $0 \cdot c_i$ can be omitted

"shopping lists"
 Post-McAtem: $(t,0) + (0,t) \neq (t,t)$

3. repeating ones can be merged

Can add! $(2C_1 + 3C_2) + (C_4 + C_1 - 3C_2) = \dots$

Can multiply by integers!

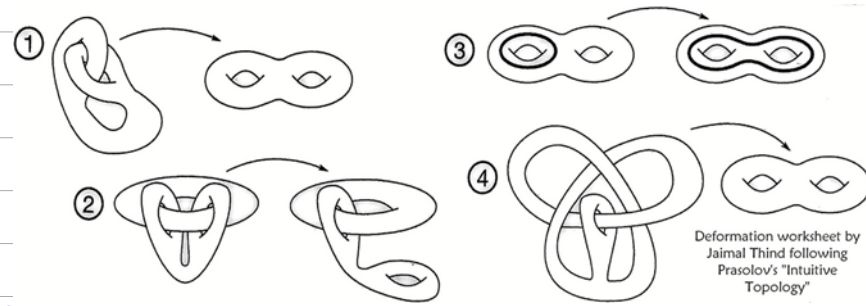
Has inverses! (Has "0")

$$I_{(j, \alpha)}^k : I^{k-1} \rightarrow I^k \quad C_{(j, \alpha)} := C \circ I_{(j, \alpha)}^k \quad \text{"free"}$$

$$\partial C := \sum_{j=1}^k (-1)^{j+k} C_{(j, \alpha)} \quad \text{Extends to } k\text{-chains!}$$

Thm $\partial^2 = 0$

Riddle Along: A function $f: \mathbb{Z}^3 \rightarrow \mathbb{R}$ is superharmonic if for every x , $f(x)$ is greater than or equal to the average of f on the 6 points adjacent to x . Is there a non-negative, non-constant, superharmonic function on \mathbb{Z}^3 ?



$$d: \Omega^k \rightarrow \Omega^{k+1} \text{ by } dw = \sum_{i=1}^n dx_i \wedge \frac{\partial w}{\partial x_i}$$

$df(\xi) = D_\xi f$, linear, (super)-Leibnitz,

compatible w/ pullbacks, $d^2 = d \circ d = 0$, satisfies "infinitesimal

Aside closed & exact, Poincaré's Lemma,

Forms on an open set, $\lambda = \frac{-y dx}{x^2 + y^2} + \frac{x dy}{x^2 + y^2}$

$$\int_P dw = \int_{\partial P} w$$

Singular k -cube in $A \subset \mathbb{R}^n$: Cont. $C: [0,1]^k \rightarrow A$

(0-cubes, 1-cubes)

(The space of k -chains in A) := The free Abelian group generated by all k -cubes = $\left\{ \sum_{i=1}^m a_i c_i \right\}$

"shopping lists"
"inventories"

3. repeating entries can be merged

1. order immaterial
2. $0 \cdot c_i$ can be omitted

Can add! $(2c_1 + 3c_2) + (c_4 + c_1 - 3c_2) = \dots$

Can multiply by integers!

Has inverses! (Has "0")

Post-Mortem:
 $(0,t) + (t,0) \neq (t,t)$

$I_{(j,\alpha)}^k : I^{k-1} \rightarrow I^k$ $C_{(j,\alpha)} := C \circ I_{(j,\alpha)}^k$ "free"

$\partial C := \sum_{j=1}^k (-1)^{j+1} C_{(j,\alpha)}$ Extends to k -chains!

Thm $\partial^2 = 0$. Lemma $(C_{(j,\alpha)})_{(j,\beta)} = (C_{(j+1,\beta)})_{(j,\alpha)}$ if $k \leq j$

$$C_k(A) = (\text{k-chains in } A) = \left\{ \sum_{i=1}^m a_i c_i : c_i: I^k \rightarrow A \text{ cont. } \boxed{\text{diffable/smooth}} \right\} / \text{shopping list rules}$$

An Abelian group / a \mathbb{Z} -module

$$I_{(j,\alpha)}^k: I^{k-1} \rightarrow I^k \text{ by } (x_1, \dots, x_{k-1}) \mapsto (x_1, \dots, \alpha_j, x_{j+1}, \dots, x_k)$$

$$C_{(j,\alpha)} := C \circ I_{(j,\alpha)}^k \quad \text{"face"} \quad \partial C := \sum_{i=1}^k \sum_{\alpha \in \{0,1\}} (-1)^{j+\alpha} C_{(j,\alpha)}$$

Thm $\partial^2 = 0$. Extends to k-chains!

Lemma $(C_{(j,\alpha)})_{(j,\beta)} = (C_{(j+1,\beta)})_{(j,\alpha)}$ if $k \leq j$

PF of thm

Push or pull? Compatibility w/ ∂

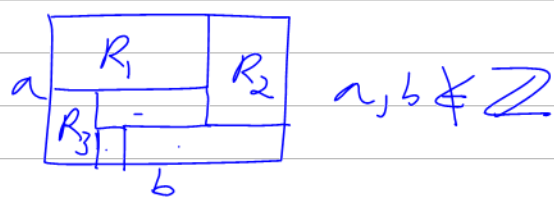
$$\int_{I^k} f dx_1 \wedge \dots \wedge dx_k \quad \text{No examples!}$$

$$A \subset \mathbb{R}^n, C \in C_k(A), W \in \Omega^k(A), \int_C W$$

Example $C = \begin{pmatrix} \cos 2\pi\theta \\ \sin 2\pi\theta \end{pmatrix} \quad W = \frac{-y dx}{x^2+y^2} + \frac{x dy}{x^2+y^2}$

example Suppose $C: I^2 \rightarrow \mathbb{R}^2_{xy}$ is 1-1 and w/ $\det(C') > 0$ & $A = C(I^2)$ Then $\int_C dx \wedge dy = \dots$

Compatibility of push forwards & pull backs



$$I_{(j,x)}^k : I^{k-1} \rightarrow I^k \text{ by } (x_1, \dots, x_{k-1}) \mapsto (x_1, \dots, x_j, x_{j+1}, \dots, x_k)$$

$$C_{(j,x)} = C \circ I_{(j,x)}^k \text{ "face" } \quad \partial C := \sum_{j=1}^k \sum_{x \in \{0,1\}} (-1)^{j+x} C_{(j,x)}$$

$$\int_{I^k} F dx_1 \wedge \dots \wedge dx_k = \int_{I^k} F \quad \partial^2 = 0$$

$$A \subset \mathbb{R}^n, \quad C \in C_k(A), \quad w \in \Omega^k(A), \quad \int_C w := \int_{I^k} C^* w$$

example $C = \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \end{pmatrix} \quad w = \frac{-y dx}{x^2 + y^2} + \frac{x dy}{x^2 + y^2}$

[by Stokes' thm, w is not exact]

example Suppose $C: I^2 \rightarrow \mathbb{R}_{xy}^2$ is 1-1 curve w/ $\det(C') > 0$ & $A = C(I^2)$. Then $\int_C dx \wedge dy = \dots$

Push or pull? Compatibility w/ ∂ & w/ \int

Reminder HW15Q1 $\Omega^0(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3)$

Thms $C \in C_k(A), w \in \Omega^{k-1}(A), A \subset \mathbb{R}^n$

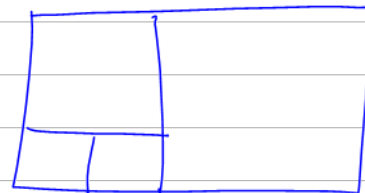
$$\text{then } \int_C dw = \int_{\partial C} w$$

PF WLOG, $C: I^k \rightarrow A$ is a single curve.

$$\int_C dw = \int_{I^k} C^*(dw) = \int_{I^k} d(C^*w) \stackrel{?}{=} \int_{\partial I^k} C^*w = \int_{\partial C} w = \dots$$

then do I^k , $w = F dx_1 \wedge \dots \wedge dx_k$

Riddle Along: Can you partition a rectangle exactly one of whose sides is irrational into finitely many squares?



A Question for Deep Reflection: Mirrors flip left and right, but not up and down. How can mirrors tell horizontal from vertical?

$$(x_1, \dots, x_{k-1}) \xrightarrow{I_{(j,k)}} (x_1, \dots, x_j, x_{j+1}, \dots, x_k) \quad \partial C = \sum_{j=1}^k \sum_{\alpha \in \{0,1\}} (-1)^{j+\alpha} C \circ I_{(j,\alpha)}^k$$

$$C \in C_k(A \subset \mathbb{R}^n), w \in \Omega^k(A) \quad \int_C w := \int_{I^k} C^* w := \int_{I^k} F dx_1 \dots dx_k$$

Thm $C \in C_k(A \subset \mathbb{R}^n), w \in \Omega^{k-1}(A) \Rightarrow \int_C dw = \int_C w$

PF WLOG, $C: I^k \rightarrow A$ is a single cube.

$$\int_C dw = \int_{I^k} C^*(dw) = \int_{I^k} d(C^*w) \stackrel{?}{=} \int_{I^k} C^*w = \int_C w = \dots$$

then do I^k , $w = F dx_1 \dots dx_{k-1}$, $F dx_{k-1}$

(use y_i for the variables inside faces)

Post Mortem: I should have prepared in advance the "middle term" of lhs = ... = rhs.

Exercise Given $C: I^k \rightarrow A \subset \mathbb{R}^n$, $w \in \Omega^k(A)$ and

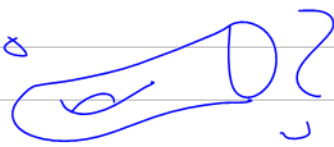
$$r: I^k \rightarrow I^k \text{ 1-1, onto, w/ } \det r' > 0, \quad \int_C w = \int_{C \circ r} w$$

(show Spivak P 104) *done link*

Exercise Given $C: I^k \rightarrow A \subset \mathbb{R}^n$, $\omega \in \Omega^k(A)$ and $r: I^k \rightarrow I^k$ 1-1, onto, w/ $\det r' > 0$, $\int_C \omega = \int_{C \circ r} \omega$

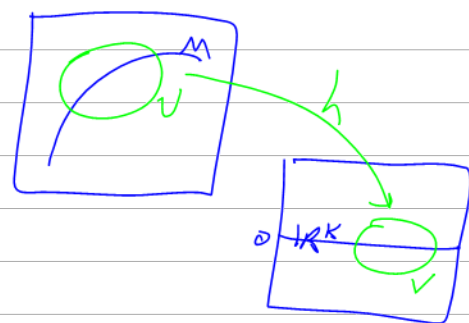
So integration of forms depends only on the images of cubes. Can we extend it to

Thm Given $k \leq n$, $M \subset \mathbb{R}^n$, $x \in M$, TFAE:



(M) \exists open $U \ni p$, open $V \subset \mathbb{R}^k$

& a diffeomorphism $h: U \rightarrow V$ s.t. smooth w/ smooth inverse



$$h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\})$$

(Z) \exists open $U \ni p$ & smooth $g: U \rightarrow \mathbb{R}^{n-k}$

s.t. $U \cap M = U \cap g^{-1}(0)$ & $\text{rank}(g') = n-k$

(C) \exists open $U \ni p$, open $W \subset \mathbb{R}^k$ & smooth 1-1

$F: W \rightarrow \mathbb{R}^n$ s.t. ① $F(W) = M \cap U$

② $F^{-1}: M \cap U \rightarrow W$ is cont. ③ $\forall a \in W, \text{rank } F'_a = k$

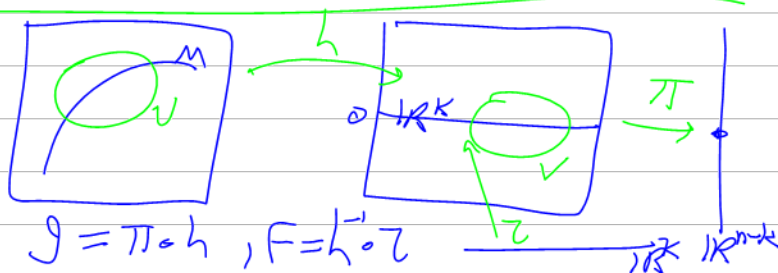
Def of a manifold, S^1 in two ways, $S^1 = \{x \in \mathbb{R}^{n+1} : |x|=1\}$

$$T^2 = \text{torus} = \left\{ \begin{pmatrix} (2+\cos\phi)\cos\theta \\ (2+\cos\phi)\sin\theta \\ \sin\phi \end{pmatrix} \right\}$$

PF of thm

$M \Rightarrow Z, C$: Trivial

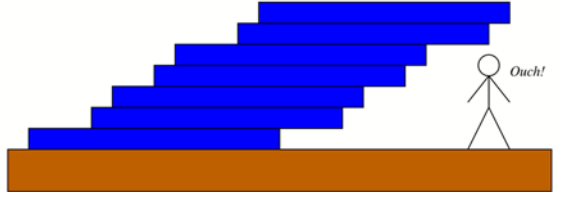
$C, Z \Rightarrow M$: next class



Within 10 minutes of the start of the test yesterday, the questions were posted on a question-sharing web site. Somebody thinks they are smart! We'll work to prove them wrong.

There was an issue with Q4; it will be managed after I know how many students were affected.

How far sideways can you reach by stacking up n identical blue domino pieces, before your tower will lean over and fall? What if n goes to infinity? (no glue is allowed, and as shown, the stacking isn't necessarily "even")

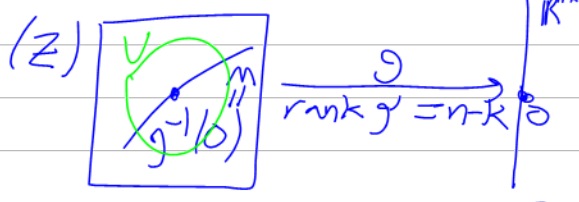
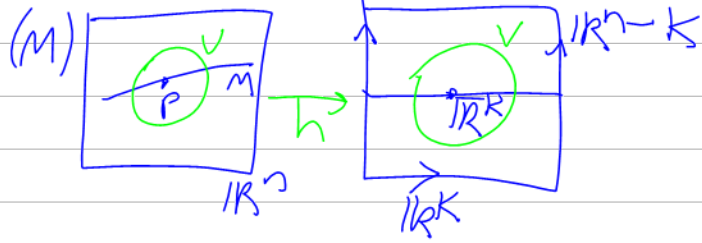


Thm Given $K \subseteq \mathbb{R}^n$, $M \subset \mathbb{R}^n$, $x \in M$, TFAE:

(M) \exists open $U \ni p$, open $V \subset \mathbb{R}^{n-k}$ & a diffeomorphism $h: U \rightarrow V$ s.t. $h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\})$.

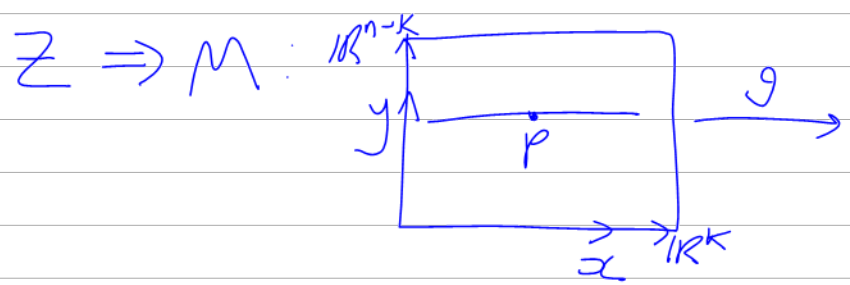
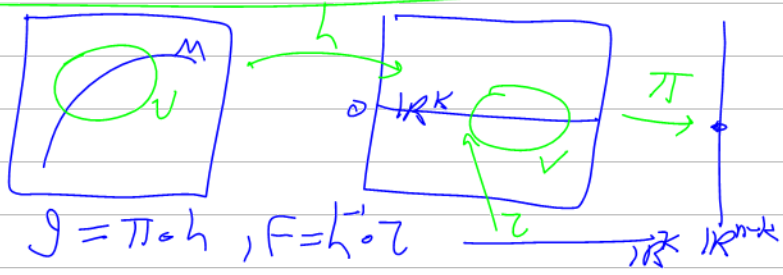
(Z) \exists open $U \ni p$ & smooth $g: U \rightarrow \mathbb{R}^{n-k}$ s.t. $U \cap M = U \cap g^{-1}(0)$ & $\text{rank}(g') = n-k$

(C) \exists open $U \ni p$, open $W \subset \mathbb{R}^k$ & smooth 1-1 $F: W \rightarrow \mathbb{R}^n$ s.t. ① $F(W) = M \cap U$
 ② $F^{-1}: M \cap U \rightarrow W$ is cont. ③ $\forall a \in W, \text{rank}(F'(a)) = k$.



PF of thm

$M \Rightarrow Z, C$: Trivial



WLOG,
 $\text{rank} \frac{\partial g}{\partial y} = n-k$

set $h(x,y) = (x, g(x,y))$

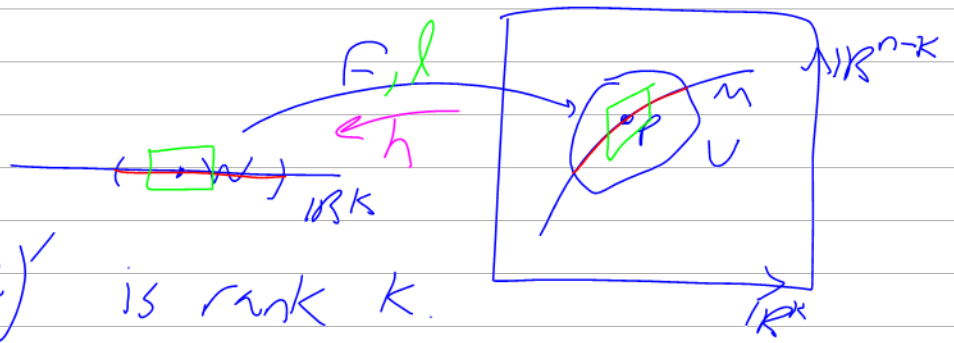
$$h'(p) = \begin{pmatrix} I & 0 \\ * & \frac{\partial g}{\partial y} \end{pmatrix}$$

So use the IFT.

Aside. In C, cond ② is necessary:



$$C \Rightarrow M$$



wlog, $(\pi_1 \circ f)'$ is rank k .

$$\text{set } l(x, y) = F(x) + \begin{pmatrix} 0 \\ y \end{pmatrix} \quad l' = \begin{pmatrix} \frac{\partial f_1}{\partial x} & 0 \\ \frac{\partial f_2}{\partial x} & I \end{pmatrix}$$

NTS. For smaller U, V

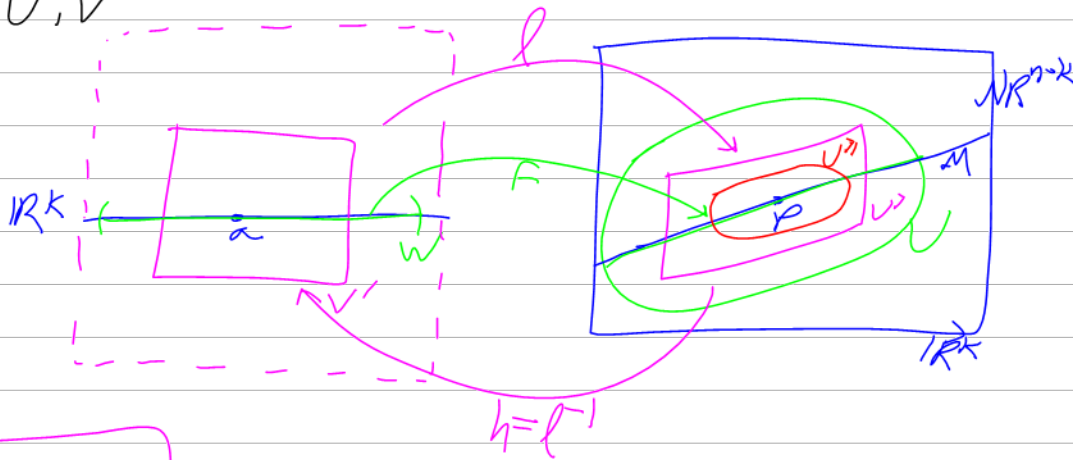
$$h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\})$$

\Downarrow

$$l(V \cap \mathbb{R}^k) = U \cap M$$

\Downarrow

$$F(V \cap \mathbb{R}^k) = U \cap M$$



Aside $\mathbb{R}^n \rightarrow \mathbb{R}^m$
 $F: A \rightarrow B$ cont

$\Leftrightarrow \exists \tilde{V} \subset \mathbb{R}^m$ open $\Rightarrow \exists$ open \tilde{U}
 in \mathbb{R}^n s.t. $F^{-1}(\tilde{V}) = A \cap \tilde{U}$

$F^{-1}: M \cap U \rightarrow W$ cont
 $F: W \rightarrow M \cap U$ is open
 $F(V \cap \mathbb{R}^k)$ is open in $M \cap U$
 so it is $M \cap U$

Read Along: Spivak 109-115.

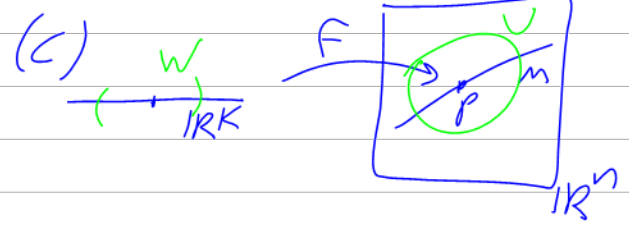
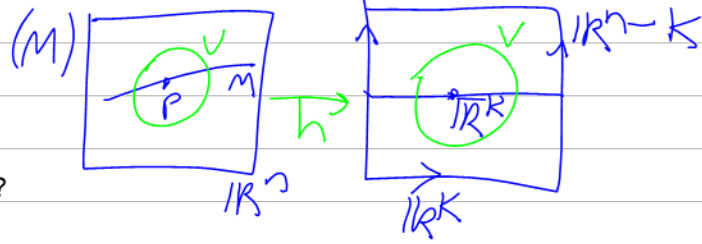
No news yet on TT3.

Our final assessment: Tuesday April 20 9-12 (morning TO time).

Old riddle (sol'n at end): Can you fit 125 4x2x1 boxes in one 10x10x10 cube?

Thm Given $k \leq n$, $M \subset \mathbb{R}^n$, $p \in M$, $(C) \Rightarrow (M)$

(M) \exists open $U \ni p$, open $V \subset \mathbb{R}^k$
 & a diffeomorphism $h: U \rightarrow V$ s.t.
 $h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\})$



(C) \exists open $U \ni p$, open $W \subset \mathbb{R}^k$ & smooth 1-1 $F: W \rightarrow \mathbb{R}^n$ s.t.

- ① $F(W) = M \cap U$
- ② $F^{-1}: M \cap U \rightarrow W$ is cont.
- ③ $\forall a \in W$, $\text{rank } F(a) = k$.

PF For convenience, assume $(\pi_1 \circ F)' = (\pi_1 \circ F)'$ is rank k

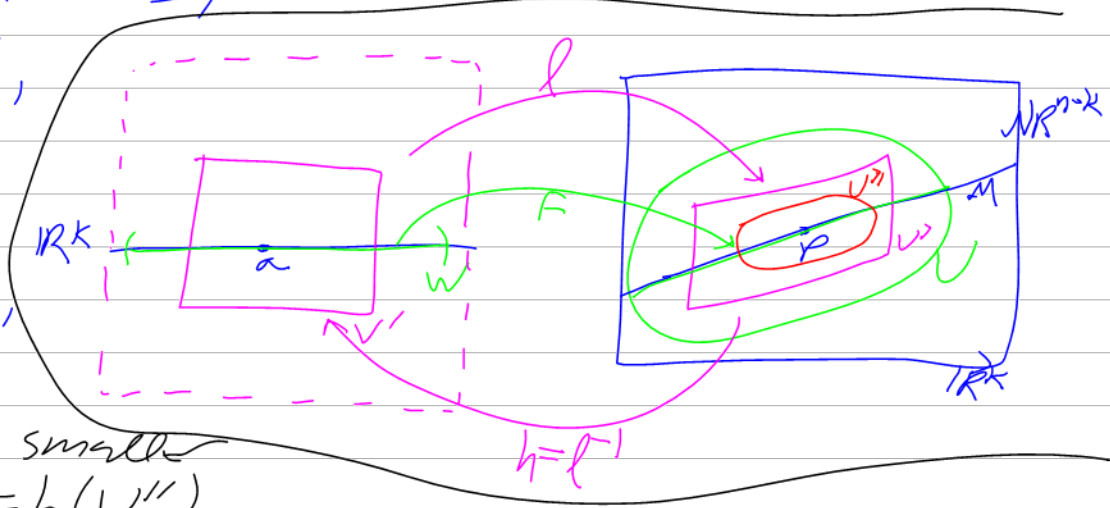
Let $l: W \times \mathbb{R}^{n-k} \rightarrow \mathbb{R}^n$ be $(x, y) \mapsto F(x) + (y)$

then $l' = \begin{pmatrix} \partial F / \partial x & 0 \\ \partial F / \partial x & I \end{pmatrix}$ is invertible, so so is

$l|_{V'}: V' \rightarrow U'$,

where $V' \subset W \times \mathbb{R}^{n-k}$
 & $U' \subset U$

Let $h = l^{-1}: U' \rightarrow V'$



N.T.S. For an even smaller $U'' \subset U'$, w/ $V'' = h(U'')$

$$h(U'' \cap M) = V'' \cap (\mathbb{R}^k \times \{0\}) \Leftrightarrow l(V'' \cap \mathbb{R}^k) = U'' \cap M \Leftrightarrow F(V'' \cap \mathbb{R}^k) = U'' \cap M$$

Aside $\mathbb{R}^n \rightarrow \mathbb{R}^m$
 $F: A \rightarrow B$ cont

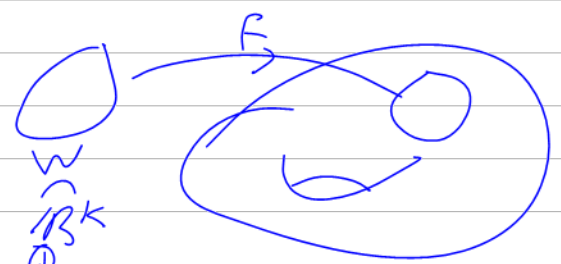
$F^{-1}: M \cap U \rightarrow W$ cont

$\Leftrightarrow \tilde{V} \subset \mathbb{R}^m$ open $\Rightarrow \exists$ open \tilde{U}
 in \mathbb{R}^m s.t. $F^{-1}(\tilde{V}) = A \cap \tilde{U}$

$F: W \rightarrow M \cap U$ is open
 $F(V' \cap \mathbb{R}^k)$ is open in $M \cap U$
 so it is $M \cap U''$ □

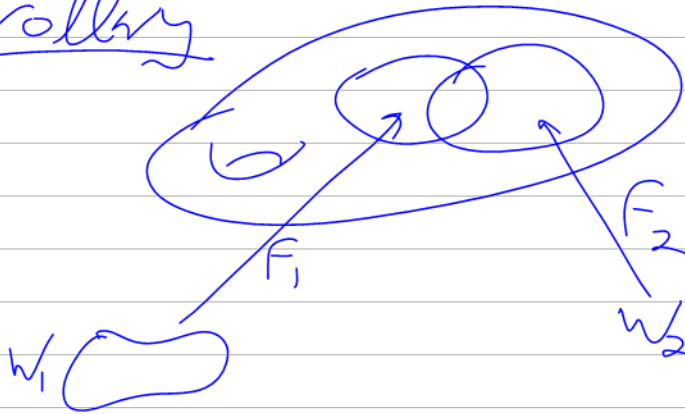
On to coord patches, Mfld w/ ∂ , ∂M , ∂M is a mflld.

Def IF $M^k \subset \mathbb{R}^n$ is a mfd,
 the F 's of (C) are called
 "coord. patches"



- ①
- ② $\forall w \in W, F'(w)$ is 1-1.
- ③

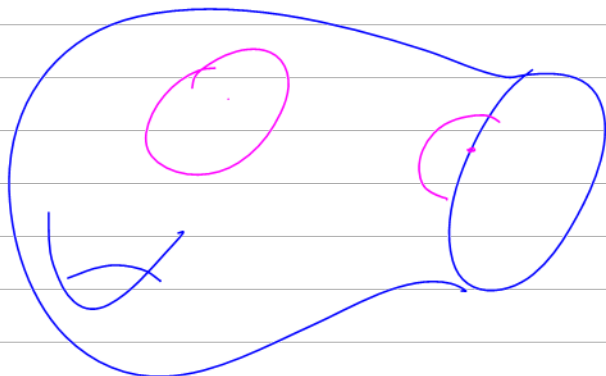
Corollary



$F_2^{-1} \circ F_1 : F_1^{-1}(W_1) \rightarrow F_2^{-1}(W_2)$
 is a diffeo.
 (smooth w/ smooth inverse)
 "transition functions"

Some philosophy on mfd's as seen in 367
 "An object is lots of partial views,
 along w/ the knowledge of how to move
 between them"

Manifold w/ bndry



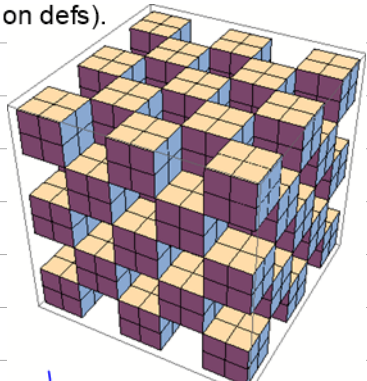
$$F: W \rightarrow M$$

$\mathbb{R}^k \xrightarrow{x \geq 0} \mathbb{R}^n$

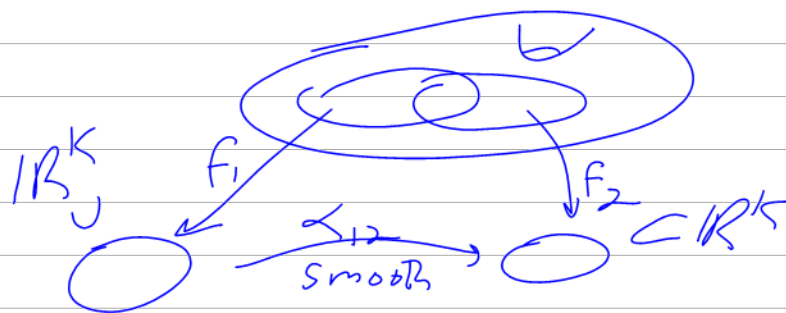
1-1 & smooth*

- ①
- ②
- ③

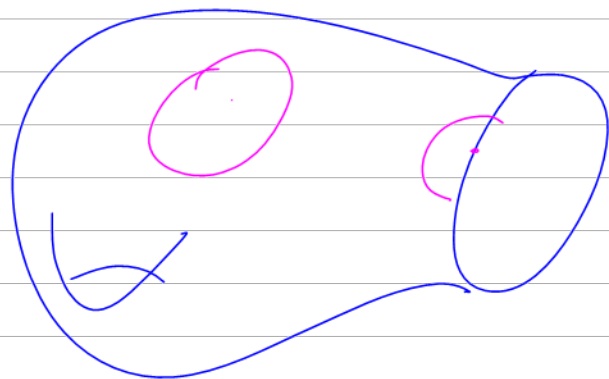
1. a pt is either "interior" or "boundary"
2. ∂M is a mfd.



shame on me!
#blue = 504



Manifold w/ bndry



$$F: W \rightarrow M$$

$\mathbb{R}^k \xrightarrow{x_k \geq 0}$ \mathbb{R}^n

1-1 & smooth*

- ①
- ②
- ③

1. a pt is either "interior" or "bndry"

2. ∂M is a mfd.

Tangent space $T_p M$ [image from a pt, idea of the pt's]

Vector Fields [smoothness]

P-Forms.

d

orientation Examples S^1, S^2, pt

Möb

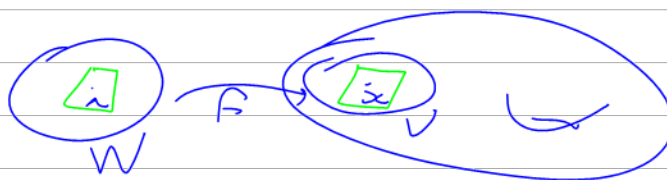
orientable, oriented.

orienting the bndry by prepending & outward pointing normal.

Old riddle (sol'n at end). The Moscow Subway Problem: Can you fit a box of dimensions $a \times b \times c$ inside a box of dimensions $a \times b \times c$, if $a+b+c > a+b+c$?

$$M_x := T_x M := F_* T_a W$$

indep of the patch!



There is $F_*: T_a W \rightarrow T_x M$, so there is $F^\#: T_x M \rightarrow T_a W$

Vector Fields [smoothness]

P-Forms. $F^*: \Omega^p(U) \rightarrow \Omega^p(W)$ an iso!

$+$, \wedge , d make sense & obey the same rules!

Example on S^2 $x dy dz + y dz dx + z dx dy$
 $x dx + y dy + z dz$

orientation Examples S^1 , S^2 , pt
 Möb

orientable, oriented.

orienting the bndry by prepping & outward pointing normal.

$$\left(\frac{x dy - y dx}{x^2 + y^2}\right) \wedge dz = \frac{\frac{x^2}{z} + \frac{y^2}{z}}{x^2 + y^2} dx dy = \frac{1}{z} dx dy$$

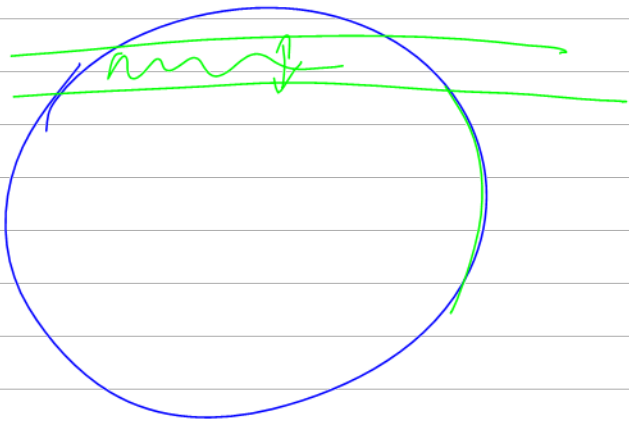
$$x dy dz + y dz dx + z dx dy = \left(\frac{x^2 + y^2}{z} + \frac{z^2}{z}\right) dx dy$$

$$x dx + y dy + z dz = 0 \Rightarrow$$

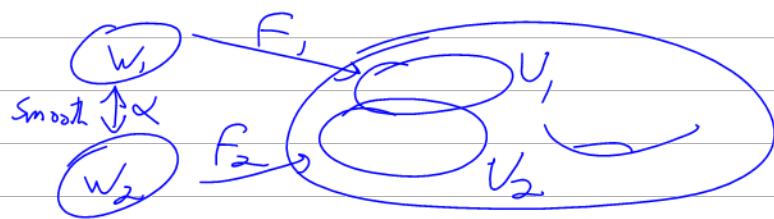
$$x dz dx = y dy dz$$

$$y dx dy = z dz dx$$

$$z dy dz = x dx dy$$



Work patch by patch, use the smooth α to show that patches agree: $T_x M, \nu^*(M), +, \wedge, d$.

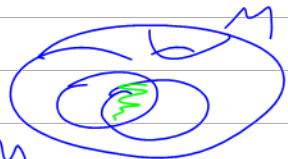


Sometimes hard to tell if $W_1 = W_2$.

Comments 1. Smooth maps $M^k \rightarrow N^l$ make sense, everything pushes & pulls under the usual rules.

2. Cubes, chains, $C^p(M)$, ∂ , \int_{chains} & Stokes \int_{M^1} all make sense.

One further ingredient is necessary to integrate on manifolds. Can we choose coord patches for the whole M (an "atlas") s.t. all trans. functions have $\det \alpha' > 0$?



Reminder orientation on V^k :

$\frac{\text{ordered basis}}{\text{pos det changes}} \sim \frac{\eta \in \Lambda^k(V)}{\text{mult by pos scalars}}$

Orientations of M: Examples S^1, S^2, pt
Mob

orientable, oriented.

orienting the boundary by prepping (& outward normal)

$\eta|_{\partial M} = i_{\partial M}^* \lrcorner_V \eta|_M$

Orientation of M^k : A choice of an orientation for each $T_x M$, which can be presented in a locally ~~continuous~~ ^{constant} manner. "orientable": has an orientation. "oriented": ^{comes with one}

Comment. If M^k is connected, it has 0 or 2 orientations.

Orienting the boundary by prepending & outward normal)

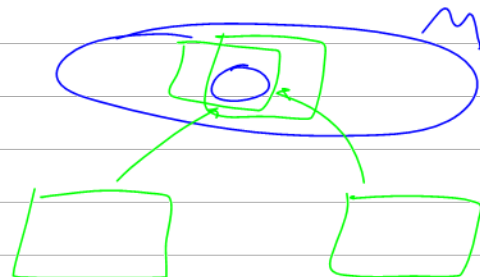
Examples D^2, D^3, I^k, D^1 roughly	$\eta_{\partial M}^{(k)} = i_{\partial M \rightarrow M}^* \lrcorner \eta_M^{(k)}$	D^1 precisely
---------------------------------------	-----------------------------------------------------------------------------------	-----------------

Proposition Let C_i ($i=1,2$) be smooth injective orientation preserving k -cubes in an oriented M^k , and let $W \in \mathcal{R}^k(M)$ be s.t.

$$\text{supp } W \subset C_1 \cap C_2$$

then $\int_{C_1} W = \int_{C_2} W$; Def call this $\int_M W$

Proof & injectivity warning




Now general integration uses

P01, assuming $\sum \int_{C_i} |W|$ is absolutely convergent

Indep. of the P01

Stokes' Theorem

Suppose $x \in \partial M \subset M$, and $\nu \in T_x M$ is the outward pointing normal to $T_x M$



then $O_{T_x M} = [\nu, u_1, \dots, u_{k-1}]$ $O_{T_x \partial M} = [u_1, \dots, u_{k-1}]$

Examples D^2, D^3, I^k , now D^1 first roughly, then

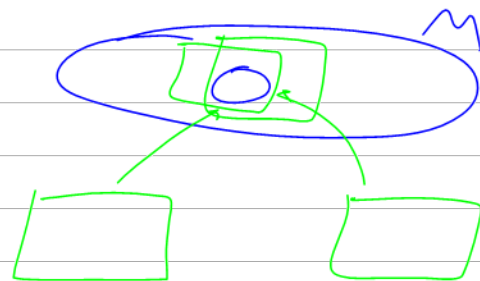
$\eta_{\partial M}^{(k)} = i_{\partial M}^* \nu \eta_M^{(k)}$, then D^1 precisely.

Proposition Let C_i ($i=1,2$) be smooth *injective* orientation preserving k -cubes in an oriented M^k , and let $W \in \Omega^k(M)$ be s.t.

$\text{supp } W \subset C_1 \cap C_2$

then $\int_{C_1} W = \int_{C_2} W$; Def Call this $\int_M W$

Proof & *injectivity warning*



Now general integration using

P01, assuming $\sum \int |W|$ is absolutely convergent

Indep. of the P01

Stokes' Theorem

Old riddle (sol'n at end). n prisoners. Each wears a tower of infinitely-many randomly-chosen b/w hats. Simultaneously each needs to point at a black hat on their head. How can they maximize the chance that they will all get it right? Can they do better than $1/2^n$?

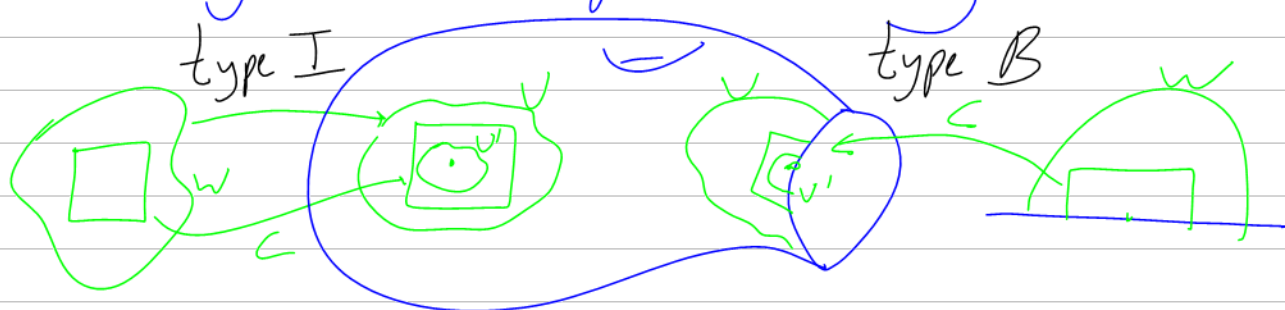
Proposition Let C_i ($i=1,2$) be smooth invertible orientation preserving k -cubes "good cubes" in an oriented M^k , and let $W \in \mathcal{L}^k(M)$ be s.t. $\text{supp } W \subset C_1 \cap C_2$

works near ∂M too!

then $\int_{C_1} W = \int_{C_2} W$; Def call this $\int_M W$

Note 1. where it makes sense, $\int_M W$ is linear in W .
 Note 2. $\int_{-M} W = -\int_M W$

Now general integration using P01.

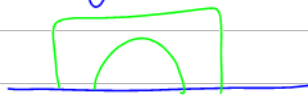
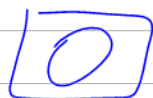


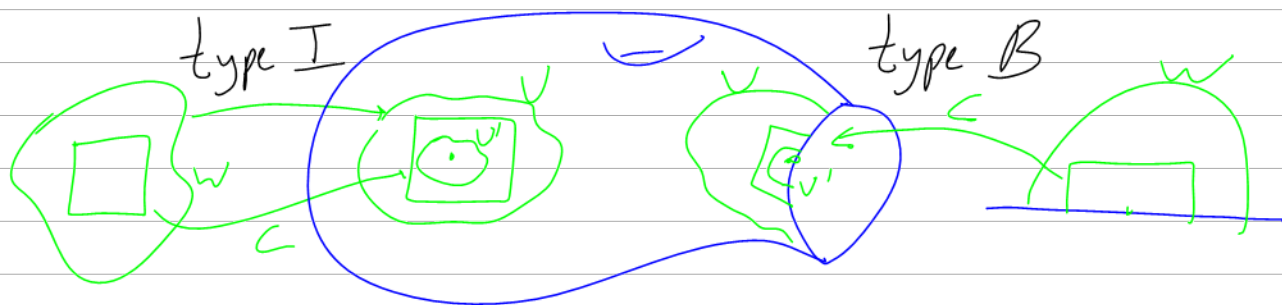
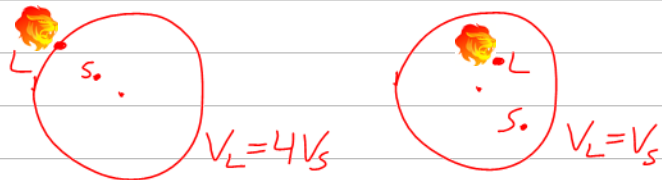
1. Assume $\sum (\varphi_i |W|)$ is absolutely convergent
2. Indep. of the P01 [automatic if M is compact]
3. The type B nbds, restricted, give cubes & P01 for ∂M .

Stokes' Thm IF M is compact and oriented &

$W \in \mathcal{L}^{k-1}(M)$, then $\int_M dW = \int_{\partial M} W$

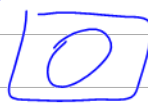
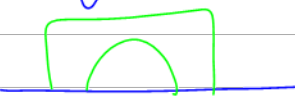
Type I Type B Then merge





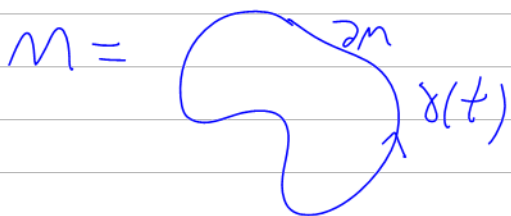
Stokes' Thm IF M is compact and oriented &

$\omega \in \mathcal{L}^{k-1}(M)$, then $\int_M d\omega = \int_{\partial M} \omega$

Type I Type B Then merge $\int_M \omega = \sum_{\partial M} \int_{\partial M} \rho_i \omega$


 $= \sum_M \int d\rho_i \omega = \sum_M (\downarrow d\rho_i) \cdot \omega + \sum_M \rho_i d\omega$

Example. The fundamental thm of calculus.


Example Green's theorem. $\omega = Pdx + Qdy$



$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

Two intuitions!

Green's theorem.



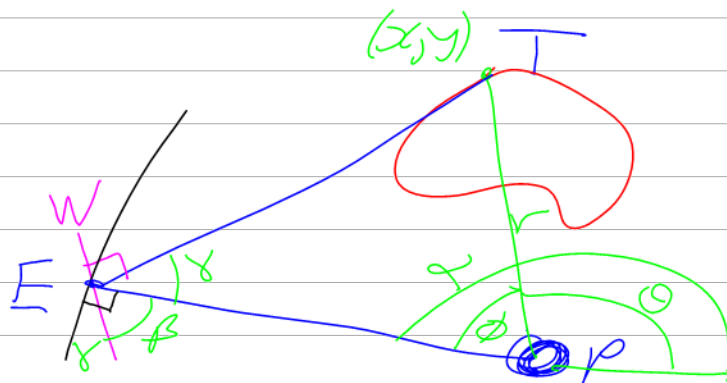
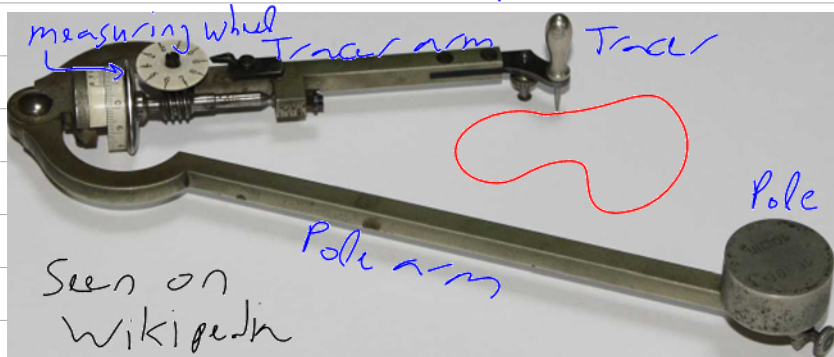
$W = Pdx + Qdy$
 $dW = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$
 (if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$,
 measures area) $\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy = \int_0^1 \dot{\gamma} \cdot \begin{pmatrix} P \\ Q \end{pmatrix} dt$



2nd interpretation $Q \rightarrow P, P \rightarrow -Q$

$$\int_D \operatorname{div} \begin{pmatrix} P \\ Q \end{pmatrix} dx \wedge dy = \int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx \wedge dy = \int_0^1 \begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{pmatrix} \cdot \begin{pmatrix} -Q \\ P \end{pmatrix} dt = \int_0^1 \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \begin{pmatrix} \dot{\gamma}_2 \\ -\dot{\gamma}_1 \end{pmatrix} dt = \int_0^1 \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \vec{n} dt$$

Example. The planimeter.



$$W = (d\alpha) \cdot \cos \gamma$$

$$= \cos(\pi - 2\phi) d(\phi + \theta) = -\cos 2\phi d(\phi + \theta)$$

$$dW = 2 \sin 2\phi d\phi \wedge d\theta = \underbrace{2 \cos \phi}_{r} \underbrace{2 \sin \phi}_{-dr} d\phi \wedge d\theta = -r dr \wedge d\theta = dx \wedge dy$$

Read Along: Spivak 126 to infinity.

HW19 is due tonight, HW20 (last!) will be online by midnight.

Old riddles (sol'n at end). Can you fold a rectangular piece of paper (perhaps many times) so that the result will have a longer perimeter than the original? If $f: A \rightarrow \mathbb{R}^2$ is a distance-non-increasing map from a rectangle to the plane, is it always the case that the length of the boundary of $f(A)$ is less than the length of the boundary of A ?

Blame COVID!



Next tasks: IF $M^3 \subset \mathbb{R}^3$ and F is a vector field,

Integration on ? on ?

From Munkres' Analysis on Manifolds:

Theorem 35.2. Let M be a compact oriented k -manifold in \mathbb{R}^n . Let ω be a k -form defined in an open set of \mathbb{R}^n containing M . Suppose that $\alpha_i: A_i \rightarrow M_i$, for $i = 1, \dots, N$, is a coordinate patch on M belonging to the orientation of M , such that A_i is open in \mathbb{R}^k and M is the disjoint union of the open sets M_1, \dots, M_N of M and a set K of measure zero in M . Then

$$\int_M \omega = \sum_{i=1}^N \left[\int_{A_i} \alpha_i^* \omega \right].$$

$$\int_M \operatorname{div} F \, dV = \int_{\partial M} F \cdot n \, dA$$

↑ unit normal to ∂M

IF $M^2 \subset \mathbb{R}^3$ is compact orient,

$$\int_M (\operatorname{curl} F) \cdot n \, dA = \int_{\partial M} F \cdot T \, ds$$

↑ unit normal to M ↑ unit tangent to ∂M

But first, what are dV , dA , ds ?

M^k oriented. dV : But multiple of the orientation

k -form for which $dV(u_1, \dots, u_k) = 1$ if (u_i) make an o.n. basis of $T_x M$

E.g. $ds(T) = 1$. $dA_{(0,0,1)} = dx \wedge dy$

In general, if $M^2 \subset \mathbb{R}^3$ & n is the positive unit normal, $dA(u, v) = \begin{vmatrix} -u \\ -v \\ -n \end{vmatrix} = (u \times v) \cdot n = \begin{matrix} n_1 dy dz \\ n_2 dz dx \\ n_3 dx dy \end{matrix}$

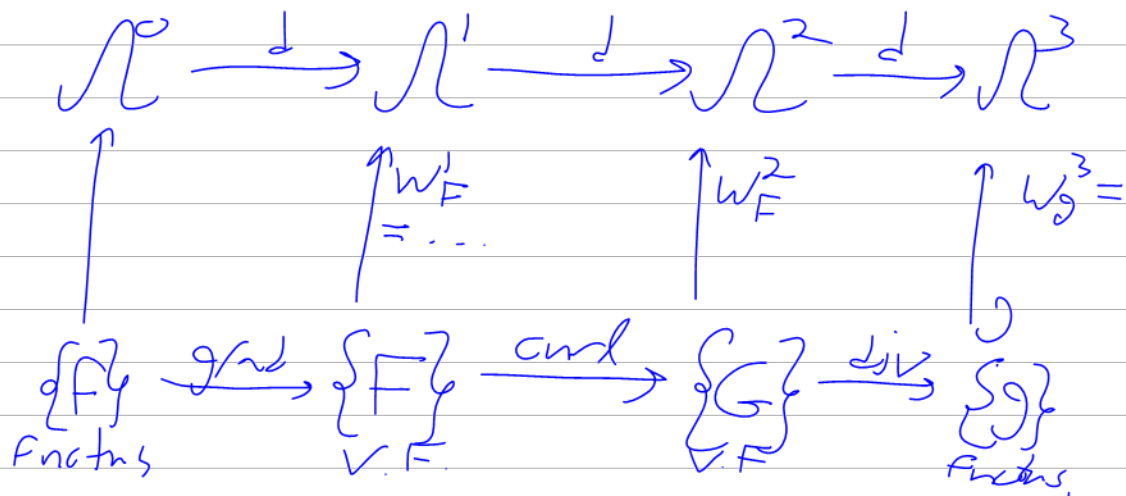
reminder: 1. Perp to u, v
2. Area
3. direction.

Example: S^2

$$= \pm |u \times v| = \pm |u| |v| \sin \theta = \pm \sqrt{|u|^2 |v|^2 - \langle u, v \rangle^2}$$

Formulas for arc length & area.

Recall
in \mathbb{R}^3



Claim 1 on $\mathcal{N}^1 \subset \mathbb{R}^3$, $W_F^1 = (T \cdot F) dS$

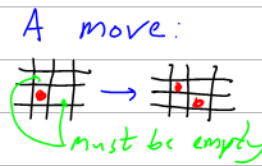
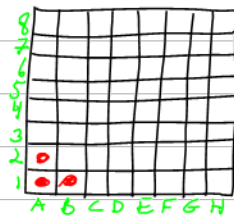
2. on $\mathcal{M}^2 \subset \mathbb{R}^3$, $W_G^2 = (G \cdot n) dA$

PF 1 Enough to compute on T .

PF 2 Enough to compute on U, V , an ON basis
of TM

Read Along: Spivak 126 to infinity.

Old riddle (sol'n at end). On a chessboard, there are three pawns at the lower left (at A1, A2, and B1). On each move, pick up one pawn, remove it and place one new pawn to the right and one new pawn above, but only if these squares are unoccupied. Can you clear the original 3 pawns?



Next tasks: IF $M^3 \subset \mathbb{R}^3$ and F is a vector field,

$$\int_M \operatorname{div} F \, dV = \int_{\partial M} F \cdot n \, dA$$

\uparrow
unit normal to ∂M

IF $M^2 \subset \mathbb{R}^3$ is compact orient,

$$\int_M (\operatorname{curl} F) \cdot n \, dA = \int_{\partial M} F \cdot T \, ds$$

\uparrow unit normal to M \uparrow unit tangent to ∂M

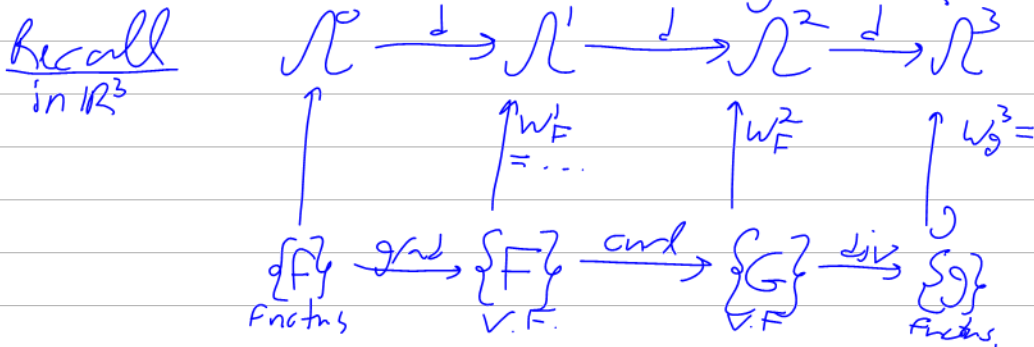
But first, what are dV , dA , ds ?

IF $M^2 \subset \mathbb{R}^3$ & n is the positive unit normal,

$$dA(u, v) = \begin{vmatrix} -u \\ -v \\ -n \end{vmatrix} = (u \times v) \cdot n = (n_1 dy dz + n_2 dz dx + n_3 dx dy)(u, v)$$

$$\Rightarrow |u \times v| = |u| |v| \sin \theta = \sqrt{|u|^2 |v|^2 - \langle u, v \rangle^2}$$

Formulas for arc length & area.



Claim 1 On $N' \subset \mathbb{R}^3$, $W_F = (T \cdot F) ds$

2 On $M^2 \subset \mathbb{R}^3$, $W_G = (G \cdot n) dA$

PF 1 Enough to compute on T

PF 2 Enough to compute on u, v , a ^{positive} ON basis of TM

$$W_G(u, v) = G \cdot (u \times v) = G \cdot n \cdot |u \times v| = G \cdot n \, dA(u, v) \quad \square$$

Thms & Geometric interpretations...

Read Along: Spivak 126 to infinity and beyond.

Note that we have a class on Monday as usual! But no Petr tutorial.

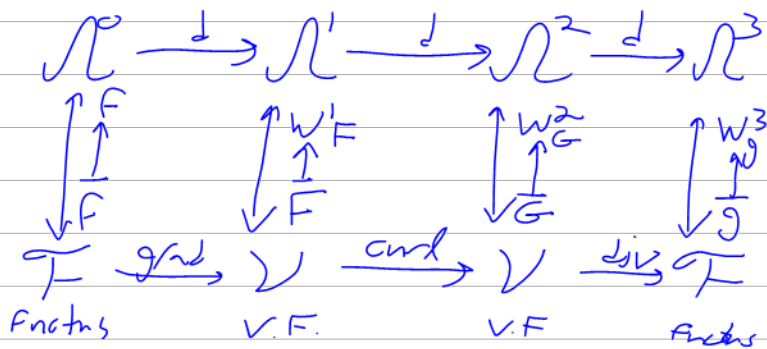
Old riddle (sol'n at end). A unit cube in \mathbb{R}^3 , the area of its projection on any plane is equal to the length of its projection on a perpendicular line to that plane.

In \mathbb{R}^3 :

1. on $N^1 \subset \mathbb{R}^3$, $W_F^1 = (T \cdot F) ds$

2. on $M^2 \subset \mathbb{R}^3$, $W_G^2 = (G \cdot n) dA$

$$\int_M dW = \int_{\partial M} W$$



M^3 , $W = W_G^2$ $dW = W_{div G}^3 = (div G) dx dy dz$

$$\int_M (div G) = \int_{\partial M} W_G^2 = \int_{\partial M} (G \cdot n) dA$$

Interpretation...

M^2 $W = W_F^1$ $dW = W_{curl F}^2$

$$\int_M dW = \int_{\partial M} W$$

$$\int_M (curl F) \cdot n dA = \int_{\partial M} (F \cdot T) ds$$

Interpretation...

Towards Maxwell

As at Maxwell-Slides@210409.pdf

