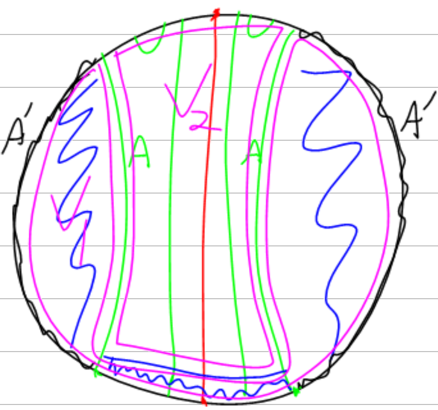
In the class of Friday Nov 27 I drew the following picture and wrote the following sentence:



One of the V_i contains p replace B by that V_i and we're down one zebra stripe.

It is wrong! Replacing B with V_i does not reduce its intersections with Σ!

The following pages are from Lickorish's "An Introduction to Knot Theory", Springer GTM 175. I hope their inclusion here falls under "fair use". I could not follow one step in Lickorish's proof. Other sources (and there are many) prove unique factorization by other means, not through Lickorish's Theorem 2.10.

Theorem 2.9. Schönflies Theorem. *Let $e : S^2 \longrightarrow S^3$ be any piecewise linear embedding. Then $S^3 - eS^2$ has two components, the closure of each of which is a piecewise linear ball.*

No proof will be given here for this fundamental, non-trivial result (for a proof see [81]). The piecewise linear condition has to be inserted, as there exist the famous "wild horned spheres" that are are examples of topological embeddings $e : S^2 \longrightarrow S^3$ for which the complementary components are not even simply connected.

The next result considers the different ways in which a knot might be expressed as the sum of other knots. It is the basic result needed to show that the expression of a knot as a sum of prime knots is essentially unique. The technique of its proof again consists of minimising the intersection of surfaces in S^3 that meet transversely in simple closed curves, but the procedure here is more sophisticated than in the proof of Theorem 2.4. In the proof, use will be made of the idea of a *ball-arc pair*. Such a pair is just a 3-ball containing an arc which meets the ball's boundary at just its two end points. The pair is unknotted if it is pairwise homeomorphic to $(D \times I, \star \times I)$, where $\star$ is a point in the interior of the disc D and I is a closed interval.

Theorem 2.10. *Suppose that a knot K can be expressed as $K = P + Q$, where P is a prime knot, and that K can also be expressed as $K = K_1 + K_2$. Then either*
(a) $K_1 = P + K_1'$ *for some* K_1', *and* $Q = K_1' + K_2$, *or*
(b) $K_2 = P + K_2'$ *for some* K_2', *and* $Q = K_1 + K_2'$.

PROOF. Let Σ be a 2-sphere in S^3, meeting K transversely at two points, that demonstrates K as the sum $K_1 + K_2$. The factorisation $K = P + Q$ implies that there is a 3-ball B contained in S^3 such that $B \cap K$ is an arc α (with K intersecting ∂B transversely at the two points $\partial\alpha$) so that the ball-arc pair (B, α) becomes, on gluing a trivial ball-arc pair to its boundary, the pair (S^3, P). As in the proof of Theorem 2.4, it may be assumed, after small movements of Σ, that Σ intersects ∂B transversely in a union of simple closed curves disjoint from K. The immediate aim will be to reduce $\Sigma \cap \partial B$. Note that if this intersection is empty, then B is contained in one of the two components of $S^3 - \Sigma$, and the result follows at once.

As $\Sigma \cap K$ is two points, any oriented simple closed curve in $\Sigma - K$ has linking number zero or ± 1 with K. Amongst the components of $\Sigma \cap \partial B$ that have zero linking number with K select a component that is innermost on Σ (with $\Sigma \cap K$ considered "outside"). This component bounds a disc $D \subset \Sigma$, with $D \cap \partial B = \partial D$. Now ∂D bounds a disc $D' \subset \partial B$ with $D' \cap K = \emptyset$ (by linking numbers), though $D' \cap \Sigma$ may have many components (see Figure 2.5). By the Schönflies theorem, the sphere $D \cup D'$ bounds a ball. "Moving" D' across this ball to just the other side of D changes B to a new position, with $\Sigma \cap \partial B$ now having fewer components than before. As the new position of B differs from the old by the addition or subtraction of a ball disjoint from K, the new (B, α) pair corresponds to P exactly as before. After repetition of this procedure, it may be assumed that each component of $\Sigma \cap \partial B$ has linking number ± 1 with K. (Thus, on each of the spheres Σ and ∂B, the components of $\Sigma \cap \partial B$ look like lines of latitude encircling, as the two poles, the two intersection points with K.)

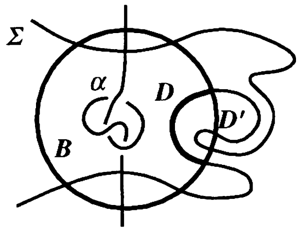


Figure 2.5

If now $\Sigma \cap B$ has a component that is a disc D, then $D \cap K$ is one point, and as P is prime, one side of D in B is a trivial ball-arc pair (see Figure 2.5). Removing from B (a regular neighbourhood of) this trivial pair produces a new B with the same properties as before but having fewer components of $\Sigma \cap B$. Thus it may be assumed that every component of $\Sigma \cap B$ is an annulus.

Let A be an annulus component of $\Sigma \cap B$. Then ∂A bounds an annulus A' in ∂B and A may be chosen (furthest from α) so that $A' \cap \Sigma = \partial A'$. Let M be the part of B bounded by the torus $A \cup A'$ and otherwise disjoint from $\Sigma \cup \partial B$. Let Δ be the closure of one of the components of $\partial B - A'$. Then Δ is a disc, with $\partial \Delta$ one of the components of $\partial A'$, and $\Delta \cap K$ equal to a single point (though $\Delta \cap \Sigma$ may have many components). This is illustrated schematically in Figure 2.6. Let $N(\Delta)$ be a small regular neighbourhood of Δ in the closure of $B - M$. This should be thought of as a thickening of Δ into $B - M$. The pair $(N(\Delta), N(\Delta) \cap \alpha)$ is a trivial ball-arc pair. However, $M \cup N(\Delta)$ is a ball, because its boundary is a sphere, and the fact that P is prime implies that the ball-arc pair $(M \cup N(\Delta), N(\Delta) \cap \alpha)$ is either trivial or a copy of the pair (B, α). If it is trivial (that is, when M is a solid torus), B may be changed, as before, by removing (a neighbourhood of) this pair to give a new B with fewer components of $\Sigma \cap B$. Otherwise, M is a copy of B less a neighbourhood of α, and that is just the exterior of the knot P; $\partial \Delta$ corresponds to a meridian of P. The closure of one of the complementary domains of Σ in S^3,

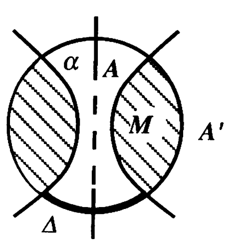


Figure 2.6

I could not recover this.

way that corresponding to K_1, contains M, and $M \cap \Sigma = A$. The meridian $\partial\Delta$ bounds a disc in $\Sigma - A$ that meets K at one point. This means that P is a summand of K_1 as required, so $K_1 = P + K_1'$ for some K_1'.

In this last circumstance, remove the interior of M and replace it with a solid torus $S^1 \times D^2$. Glue the boundary of the solid torus to ∂M, and ensure that the boundary of any meridional disc of $S^1 \times D^2$ is identified with a curve on ∂M that cuts $\partial\Delta$ at one point. Then $(S^1 \times D^2) \cup N(\Delta)$ is a ball, so B has been changed to become a new ball B', and (B', α) is a trivial ball-arc pair. The closure of $S^3 - B$ is unchanged; it is still a ball, so S^3 is changed to a new copy of S^3. In that new copy, the knot has become Q and, viewed as being decomposed by Σ, it has become $K_1' + K_2$. Thus $Q = K_1' + K_2$. □

Corollary 2.11. *Suppose that P is a prime knot and that $P + Q = K_1 + K_2$. Suppose also that $P = K_1$. Then $Q = K_2$.*

PROOF. By Theorem 2.10, there are two possibilities. The first is that for some K_1', $P + K_1' = K_1 = P$ and $Q = K_1' + K_2$. But then the genus of K_1' must be zero, so K_1' is the unknot and so $Q = K_2$. The second possibility is that for some K_2', $P + K_2' = K_2$ and $Q = K_2' + K_1$. But then $Q = K_2' + P = K_2$. □

Theorem 2.12. *Up to ordering of summands, there is a unique expression for a knot K as a finite sum of prime knots.*

PROOF. Suppose $K = P_1 + P_2 + \cdots + P_m = Q_1 + Q_2 + \cdots + Q_n$, where the P_i and Q_i are all prime. By the theorem, P_1 is a summand of Q_1 or of $Q_2 + Q_3 + \cdots + Q_n$, and if the latter, then it is a summand of one of the Q_j for $j \geq 2$, by induction on n. Of course if P_1 is a summand of Q_j, then $P_1 = Q_j$. By the corollary, P_1 and Q_j may then be cancelled from both sides of the equation, and the result follows by induction on m. Note that this induction starts when $m = 0$. Then $n = 0$ because the unknot cannot be expressed as a sum of non-trivial knots (again by consideration of genus). □

The theorems of this chapter are intended to make it reasonable to restrict attention to prime knots in most circumstances. Certainly that is the tradition when considering knot tabulation.

Exercises

1. Prove that a non-trivial torus knot is prime by considering the way in which a 2-sphere, meeting the knot at two points, would cut the torus that contains the knot.
2. For a 2-bridge knot K there is a 2-sphere separating S^3 into two balls, each of which intersects K in two standard arcs. By considering how this sphere might intersect a 2-sphere meeting the knot at two points, prove that a non-trivial 2-bridge knot is prime.