Class of November 2: A Quick Introduction to Feynman Diagrams

We wish to understand

$$\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\,\mathrm{hol}_\gamma(A)\exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3}\mathrm{tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)\right]$$

As a warm up, suppose (λ_{ij}) is a symmetric positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

$$\int\limits_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\lambda_{ij}x^ix^j + \frac{1}{6}\lambda_{ijk}x^ix^jx^k\right)$$

$$= \int\limits_{\mathbb{R}^n} \exp\left(\frac{1}{6}\lambda_{ijk}x^ix^jx^k\right)\exp\left(-\frac{1}{2}\lambda_{ij}x^ix^j\right)$$

The Fourier Transform.

$$(f\colon V \to \mathbb{C}) \Rightarrow (\tilde{f}\colon V^* \to \mathbb{C})$$

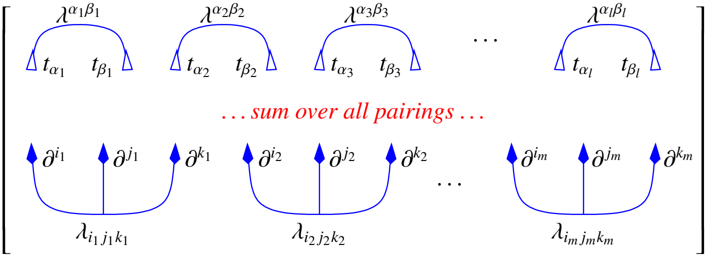
via $\tilde{f}(\varphi) := \int_V f(v)e^{-i\langle\varphi,v\rangle}dv$. Some facts:

- $\tilde{f}(0) = \int_V f(v)dv$.
- $\frac{\partial}{\partial\varphi_i}\tilde{f} \sim \widetilde{v^i f}$.
- $\widetilde{(e^{Q/2})} \sim e^{Q^{-1}/2}$, where Q is quadratic, $Q(v) = \langle Lv, v\rangle$ for $L\colon V \to V^*$, and $Q^{-1}(\varphi) := \langle\varphi, L^{-1}\varphi\rangle$. (This is the key point in one of the proofs of the Fourier inversion formula!)

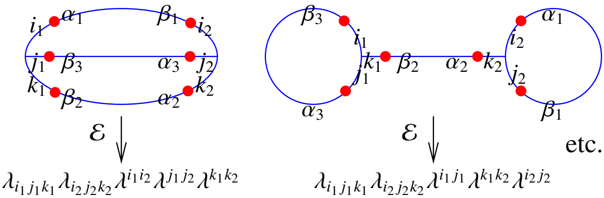
$$= C\exp\left(\frac{1}{6}\lambda_{ijk}\partial^i\partial^j\partial^k\right)\exp\left(\frac{1}{2}\lambda^{\alpha\beta}t_\alpha t_\beta\right)\Bigg|_{t_\alpha=0}$$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}} \frac{C}{6^m m! 2^l l!}\left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m\left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}} \frac{C}{6^m m! 2^l l!}$$



Examples.



$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}} \frac{C}{6^m m! 2^l l!} \sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman diagrams } D}} \mathcal{E}(D)$$