

Class of November 2: A Quick Introduction to Feynman Diagrams

We wish to understand

$$\int_{A \in \Omega^1(\mathbb{R}^3, \mathfrak{g})} \mathcal{D}A \operatorname{hol}_\gamma(A) \exp \left[\frac{ik}{4\pi} \int_{\mathbb{R}^3} \operatorname{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right]$$

As a warm up, suppose (λ_{ij}) is a symmetric positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of \mathbb{R}^n , let $(t_i)_{i=1}^n$ be a set of “dual” variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\lambda_{ij}x^i x^j + \frac{1}{6}\lambda_{ijk}x^i x^j x^k\right) \\ = \int_{\mathbb{R}^n} \exp\left(\frac{1}{6}\lambda_{ijk}x^i x^j x^k\right) \exp\left(-\frac{1}{2}\lambda_{ij}x^i x^j\right)$$

The Fourier Transform.

$$(f: V \rightarrow \mathbb{C}) \Rightarrow (\tilde{f}: V^* \rightarrow \mathbb{C})$$

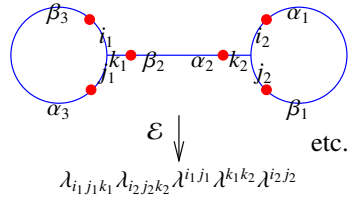
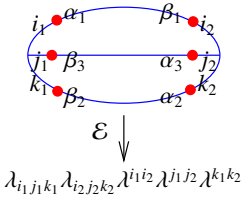
via $\tilde{f}(\varphi) := \int_V f(v) e^{-i\langle \varphi, v \rangle} dv$. Some facts:

- $\tilde{f}(0) = \int_V f(v) dv$.
- $\frac{\partial}{\partial \varphi_i} \tilde{f} \sim \widetilde{v^i f}$.
- $(\widetilde{e^{Q/2}}) \sim e^{Q^{-1}/2}$, where Q is quadratic, $Q(v) = \langle Lv, v \rangle$ for $L: V \rightarrow V^*$, and $Q^{-1}(\varphi) := \langle \varphi, L^{-1}\varphi \rangle$. (This is the key point in one of the proofs of the Fourier inversion formula!)

$$= C \exp\left(\frac{1}{6}\lambda_{ijk}\partial^i \partial^j \partial^k\right) \exp\left(\frac{1}{2}\lambda^{\alpha\beta} t_\alpha t_\beta\right) \Big|_{t_\alpha=0} \\ = \sum_{\substack{m,l \geq 0 \\ 3m=2l}} \frac{C}{6^m m! 2^l l!} (\lambda_{ijk}\partial^i \partial^j \partial^k)^m (\lambda^{\alpha\beta} t_\alpha t_\beta)^l$$

$$= \sum_{\substack{m, l \geq 0 \\ 3m = 2l}} \frac{C}{6^m m! 2^l l!} \left[\begin{array}{c} \begin{array}{ccccccc} \lambda^{\alpha_1 \beta_1} & \lambda^{\alpha_2 \beta_2} & \lambda^{\alpha_3 \beta_3} & \dots & \lambda^{\alpha_l \beta_l} \\ \begin{array}{cc} \curvearrowright & \curvearrowleft \\ t_{\alpha_1} & t_{\beta_1} \end{array} & \begin{array}{cc} \curvearrowright & \curvearrowleft \\ t_{\alpha_2} & t_{\beta_2} \end{array} & \begin{array}{cc} \curvearrowright & \curvearrowleft \\ t_{\alpha_3} & t_{\beta_3} \end{array} & \dots & \begin{array}{cc} \curvearrowright & \curvearrowleft \\ t_{\alpha_l} & t_{\beta_l} \end{array} \\ \dots \text{ sum over all pairings } \dots \\ \begin{array}{ccccccc} \uparrow \partial^{i_1} & \uparrow \partial^{j_1} & \uparrow \partial^{k_1} & \uparrow \partial^{i_2} & \uparrow \partial^{j_2} & \uparrow \partial^{k_2} & \dots & \uparrow \partial^{i_m} & \uparrow \partial^{j_m} & \uparrow \partial^{k_m} \\ \begin{array}{cc} \curvearrowright & \curvearrowleft \\ \lambda_{i_1 j_1 k_1} & \end{array} & \begin{array}{cc} \curvearrowright & \curvearrowleft \\ \lambda_{i_2 j_2 k_2} & \end{array} & \dots & \begin{array}{cc} \curvearrowright & \curvearrowleft \\ \lambda_{i_m j_m k_m} & \end{array} \end{array} \end{array} \right]$$

Examples.



$$= \sum_{\substack{m, l \geq 0 \\ 3m = 2l}} \frac{C}{6^m m! 2^l l!} \sum_{\substack{m\text{-vertex fully marked} \\ \text{Feynman diagrams } D}} \mathcal{E}(D)$$