Hour 35, Monday December 7.
Goal for the remaining two classes: More on (uvw)B. Prove that PwB has a "Taylor Expansion".
But first an apology regarding unique factorization, following
http://drorbn.net/AcademicPensieve/Classes/20-1350-KnotTheory/LickorishOnUniqueFactorization.pdf

"silly braids"

$PuB := \ker(uB \longrightarrow S)$ $\qquad PvB = \langle \sigma_{ij} : \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij},\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \rangle$

"i over j, +"

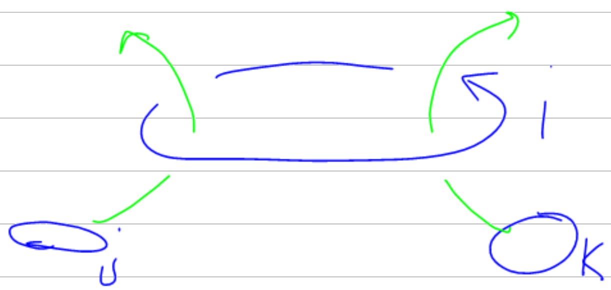
"pure virtual braids"

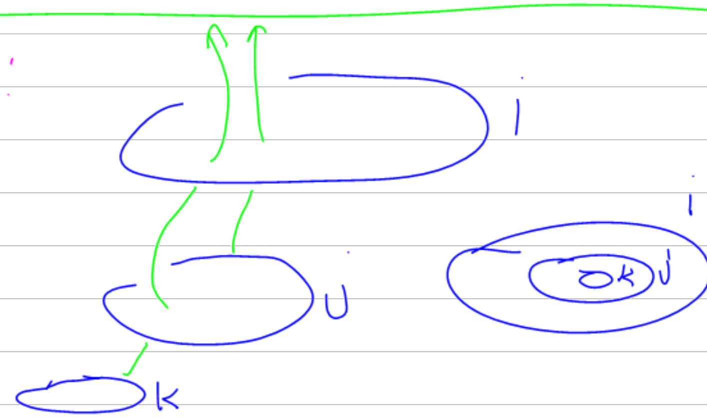
$PwB := PvB / \sigma_{ij}\sigma_{ik} = \sigma_{ik}\sigma_{ij}$

"pure w-braids" "Overcrossings commute (OC)".

σ_{ij} 

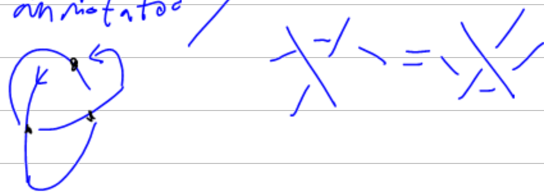
"The group of (horizontal) flying rings"

3: 

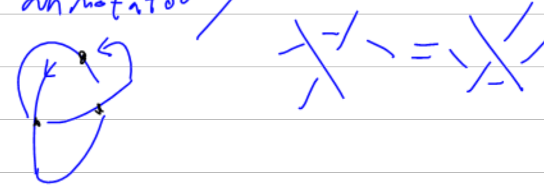
1: 

"Virtual Knots"

Knots = {planar quadrivalent graphs, appropriately annotated} / R123

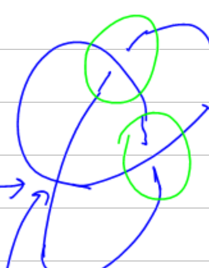


V-Knots = {quadrivalent graphs, appropriately annotated} / R123



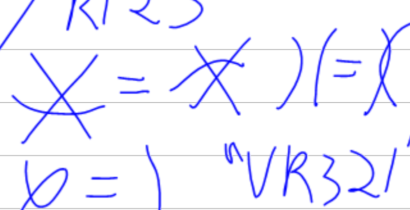
Morally-wrong equiv. def

"virtual xing"



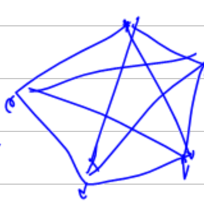
2-xing v-knots

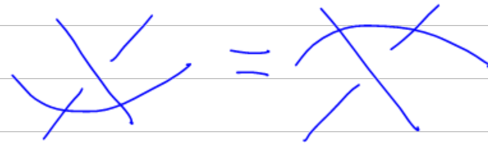
vK = {PD w/ two types of xings regular & virtual} / R123



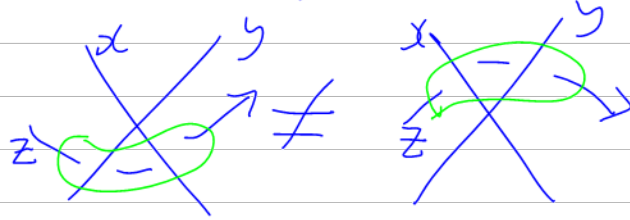
"VR321"

not a vertex a proj. artifact

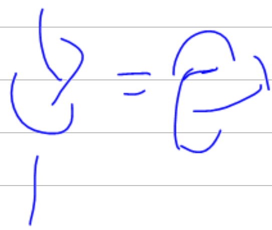


M: 

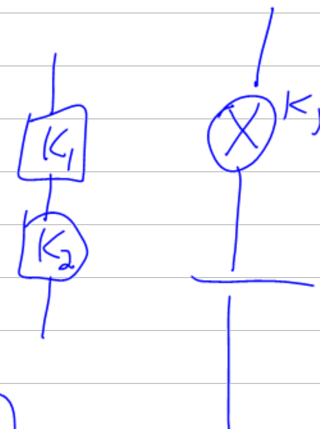
Story as z tells it
left: "I go under y then under x"
right: "I go under x then under y"

F "Forbidden" 

Some differences:

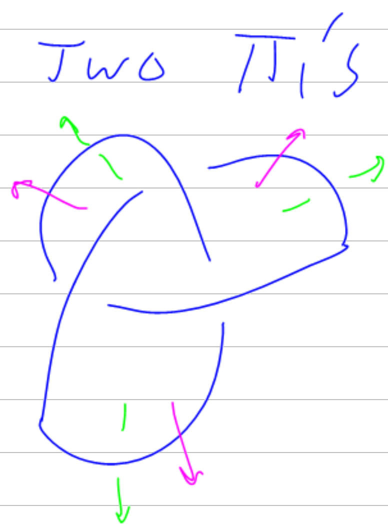
*  long v-knot $\longrightarrow$ S^1-v-knots

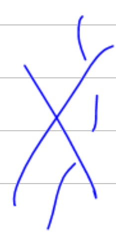
* v-K not an Abelian monoid.



* Two mirrors!

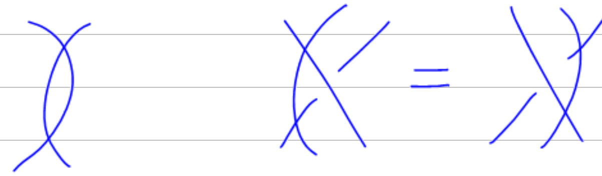
* Two π_1's





Virtual braids:

/ cls



$$vB_n = \left\langle \begin{array}{l} \sigma_i : \text{ordinary} \\ \tau_i : \text{virtual} \end{array} \right\rangle \Big/ \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j|>1 \\ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \\ \tau_i \tau_j = \ldots \\ \tau_i^2 = 1 \\ \tau_i \sigma_{i+1} \tau_i = \tau_{i+1} \sigma_i \tau_{i+1} \\ \tau_i \sigma_j = \sigma_j \tau_i \quad |i-j|>1 \end{array}$$

$$\|$$

$$PvB_n \rtimes S_n$$

Hour 36 and last, Wednesday December 9.
No time for the full story of "Taylor Expansion for PwB".
Goal: Explain what is a "Taylor Expansion" and mumble about why, for PwB, it is useful.

$$S=\langle \tau_i : \tau_i^2=1,\ \tau_i\tau_{i+1}\tau_i=\tau_{i+1}\tau_i\tau_{i+1},\ \tau_i\tau_j=\tau_j\tau_i\ \ |i-j|>1\rangle$$

$$uB=\langle \gamma_i : \gamma_i\gamma_{i+1}\gamma_i=\gamma_{i+1}\gamma_i\gamma_{i+1},\ \gamma_i\gamma_j=\gamma_j\gamma_i\ \ |i-j|>1\rangle \longleftrightarrow \underset{G/R}{\overset{\shortparallel}{PuB}}=\ker\left(B\xrightarrow{\gamma_i\to\tau_i}S\right)$$

$$\downarrow \qquad\qquad \updownarrow ?$$

$$vB=B*S\Big/\left\langle \begin{matrix}\tau_i\gamma_{i+1}\tau_i=\tau_{i+1}\gamma_i\tau_{i+1}\\ \tau_i\gamma_j=\gamma_j\tau_i\ \ |i-j|>1\end{matrix}\right\rangle \overset{?}{\longleftrightarrow} PvB=\left\langle \sigma_{ij} : \begin{matrix}\sigma_{ij}\sigma_{ik}\sigma_{jk}=\sigma_{jk}\sigma_{ik}\sigma_{ij}\\ \sigma_{ij}\sigma_{kl}=\sigma_{kl}\sigma_{ij}\end{matrix}\right\rangle$$

$$\downarrow \qquad\qquad \downarrow$$

$$\boxed{wB=vB/\gamma_i\gamma_{i+1}\tau_i=\tau_{i+1}\gamma_i\gamma_{i+1}\rangle} \overset{?}{\longleftrightarrow} \boxed{PwB=PvB/\sigma_{ij}\sigma_{ik}=\sigma_{ik}\sigma_{ij}}$$

In fact, $vB=PvB\rtimes S$ and $wB=PwB\rtimes S$ [so a "good" invariant of PwB may lead to an invt of links]

σ_{ij} 

"Taylor expansions" for groups.

Let G be a group

$\mathbb{Q}G$: the group ring of $G=\left\{\sum a_i g_i : \begin{matrix}a_i\in\mathbb{Q}\\ g_i\in G\end{matrix}\right\}$

I: the augmentation ideal in $\mathbb{Q}G$:

$$I=\left\{\sum a_i g_i : \sum a_i=0\right\}=vs\langle g_1-g_2 : g_1,g_2\in G\rangle$$

$$=\mathrm{ideal}\langle g-1 : g\in G\rangle$$

"Filtration"

$$\mathbb{Q}G=I^0\supset I^1\supset I\cdot I=I^2\supset I^3\supset I^4\supset \ldots$$

$$A(G) = \prod_{m=0}^{\infty} I^m / I^{m+1}$$

a graded ring: $\underset{\displaystyle I^k}{\overset{}{}}$ $\underset{\in I^k}{[a]}_{I^{k+1}} \cdot \underset{\in I^l}{[b]}_{I^{l+1}} = [\underset{\in I^{k+l}}{a \cdot b}]_{I^{k+l+1}}$

<u>Def</u> An expansion for a group G is a linear map

$$Z : G \longrightarrow A(G)$$

associated graded of G

st. if $a \in I^m$ then

$$Z(a) = (\underbrace{0, 0, \ldots, 0}_{m-1}, [a], *, *, *).$$

Aside take $R = C^\infty(\mathbb{R}^d)$

$$I = \{f \in R : f(0) = 0\}$$

I^m: Functions w/ zero of order m at 0.

I^m / I^{m+1}: homogeneous polynomials of deg m.

$A(R) =$ Power series around 0.

$Z : R \to A(R)$: $Z : C^\infty \longrightarrow$ Power series

HW: Verify that the Taylor expansion

is an expansion.

1. Expansion is "homomorphic" if
$Z : G \to \mathcal{A}(G)$ is multiplicative.

2. $G \to H \Rightarrow \mathcal{A}(G) \longrightarrow \mathcal{A}(H)$

$\Delta : G \to G \times G \Rightarrow \mathcal{A}(G) \xrightarrow{\Delta} \mathcal{A}(G \times G) = \mathcal{A}(G) \otimes \mathcal{A}(G)$

$\Delta : g \mapsto (g, g)$

Expansion is called co-homo if

$$\begin{tikzcd} G \arrow[r, "Z"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{A}(G) \arrow[d, "\Delta"] \\ G \times G \arrow[r, "Z \otimes Z"] & \mathcal{A}(G) \otimes \mathcal{A}(G) \end{tikzcd}$$

<u>HW 2.</u> The Taylor exp. is homo & co-homo.

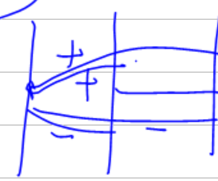
3. Any expansion $C^\infty \to$ p.s. which is both homo & co-homo is Taylor.

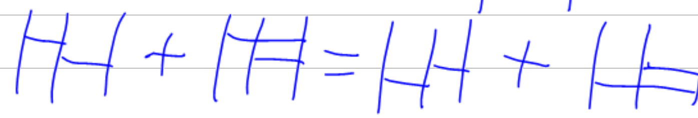
Hence one is curious which groups have Taylor expansions. . .

$G = PB$ $\qquad I_n$ $\qquad I^m$

$QG = \langle$  $\rangle$ $\quad \langle$  $\rangle$ $\quad = \langle$  $\rangle$

$\mathcal{A}(G) = \left\langle\right.$  $\left.\right\rangle$ / rels

4T: 



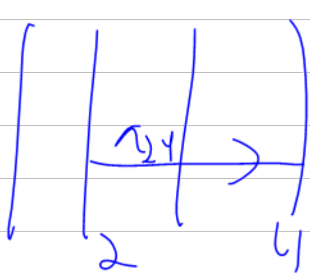
Is there a homo-co-homo expansion for ~~braids~~ PB: Yes, but it's hard.

$$PvB = \left\langle \sigma_{ij} : \begin{array}{ll} \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} & \\ \sigma_{ij}\sigma_{ik}\sigma_{jk} = \underline{\qquad} & R3 \\ \sigma_{ij}\sigma_{ik} = \sigma_{ik}\sigma_{ij} & OC \end{array} \right\rangle$$

$$\begin{array}{c} PvB \\ \parallel \\ G \end{array}$$

$I : \langle \sigma_{ij} - 1 \rangle$ in $\mathcal{A}(G) \ni [\sigma_{ij} - 1]_{I^2} = a_{ij}$

$$\mathcal{A}(G) = \left\langle a_{ij} : \begin{array}{l} [a_{ij}, a_{kl}] = 0 \\ [a_{ij}, a_{ik}] = 0 \\ 4T: [a_{ij} + a_{ik}, a_{jk}] = 0 \end{array} \right\rangle$$



$Z : G \longrightarrow \mathcal{A}(G)$

$Z(\sigma_{ij}) = e^{a_{ij}}$

$\rightsquigarrow$ Alexander poly.